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STANFORD UNIVERSITY

CENTER FOR SYSTEMS RESEARCH

Probability Densities for Diffusion Processes with Applications to Nonlinear Filtering Theory and Detection Theory

by

T. E. Duncan

May 1967

Systems Theory Laboratory

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PROBABILITY DENSITIES FOR DIFFUSION PROCESSES WITH APPLICATIONS
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ABSTRACT

Some problems in the filtering and the detection of diffusion processes that are solutions of stochastic differential equations are studied.

Transition densities for Markov process solutions of a large class of stochastic differential equations are shown to exist and to satisfy Kolmogorov's equations. These results extend previously known results by allowing the drift coefficient to be unbounded. With these results for transition densities the nonlinear filtering problem is discussed and the conditional probability of the state vector of the system conditioned on all the past observations is shown to exist and a stochastic equation is derived for the evolution in time of the conditional probability density. A stochastic differential equation is also obtained for the conditional moments. These derivations use directly the continuous time processes.

Necessary conditions that coincide with the previously known sufficient conditions for the absolute continuity of measures corresponding to solutions of stochastic differential equations are obtained. Applications are made to the detection of one diffusion process in another. Previous results on the relation between detection and filtering problems are rigorously obtained and extended.

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SYMBOLS

$a(t, x_t)$	drift vector
A^T	transpose of A
$b(t, x_t)$	diffusion matrix
$B(t, \omega)$	vector Brownian motion
$\tilde{B}(t, \omega)$	vector Brownian motion
$B(x_u, u \leq t)$	Borel σ-algebra generated by $\{x_u, u \leq t\}$
$\mathcal{B}_t$	$B(B_u, 0 \leq u \leq t)$
c	symmetric matrix
$C_n[s,t]$	n dimensional space of continuous functions on $[s,t]$
$E(\cdot)$	expectation
$E(\cdot \mid \cdot)$	conditional expectation
E_{μ_X}	expectation with respect to the measure μ_X
$\mathfrak{F}$	Borel σ-algebra on Ω
$\mathfrak{F}_t$	sub-σ-algebra of $\mathfrak{F}$
$g(t, x_t, y_t)$	drift vector
$\mathcal{G}_t$	sub-σ-algebra
$h(t, y_t)$	diffusion matrix
I_Λ	indicator function for the set Λ
$\mathcal{L}$	differential operator
L^1	space of absolutely integrable functions
L^∞	space of essentially bounded functions
M_t	Radon-Nikodym derivative
$p(s, y; t, x)$	transition density for a Markov process
p_X	transition density for $\{x_t\}$

p_Y	transition density for $\{y_t\}$
$p(\cdot \mid \cdot)$	conditional probability density
P	probability measure on Ω
$P(\cdot \mid \cdot)$	conditional probability measure
q_t	normalizing term
$r(\cdot \mid \cdot)$	unnormalized conditional probability density
$\mathcal{R}^n$	Euclidean n-space
$\mathcal{R}^+$	positive real half line, $[0,\infty)$
$x(t,\omega)$	vector stochastic process
$y(t,\omega)$	vector stochastic process
α	initial condition
$\widehat{\lambda(x_t)}$	conditional expectation of λ
Λ_t	likelihood function
μ_X	measure in function space for $\{x_t\}$
μ_Y	measure in function space for $\{y_t\}$
Π_0, Π_1	prior probabilities
ρ_X	measure in function space for $\{x_t\}$
ρ_Y	measure in function space for $\{y_t\}$
$\sigma(L^1, L^\infty)$	weak topology
φ_t	Radon-Nikodym derivative
$\Phi(t,\omega)$	vector random function
ψ_t	exponential functional
$\otimes$	product space
$\oplus$	direct sum
$\ll$	absolute continuity relation

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ACKNOWLEDGMENT

I. INTRODUCTION AND PRELIMINARIES

In this thesis we shall study some filtering and some detection problems described by stochastic processes. These problem descriptions have wide applications in physical problems because many physical phenomena can be modeled by stochastic processes.

For satellite orbit tracking and prediction problems, filtering has been effectively used to obtain "good" estimates of the satellite orbits from the noisy data received from the satellites by the ground stations. Missile and satellite guidance problems typically involve noisy measurements from the various sensors and filtering theory has been useful in improving guidance performance.

Many communication problems involve a signal corrupted by noise. This signal corruption can occur, for example, by the thermal noise in transmitters and receivers or by the properties of the medium through which the signal is transmitted. To obtain a "good" estimate of the signal, the received data must be filtered. A particular type of communication problem is feedback communication, for example, the communication from a ground station to a satellite and back to the ground station embodies the feedback principle. Filtering can be shown to provide a scheme to use this feedback communication channel in an optimal manner.

Chemical processes can often be modeled by stochastic processes where noisy measurements of the operations are obtained and filtering theory can be used to obtain "good" estimates of the operations. Some interest has developed for applying filtering techniques to models of economic systems which include random behavior. Identification problems where some of the system parameters are random can be solved by applying

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filtering theory to obtain "good" estimates of these random parameters.

In many stochastic optimal control problems the state vector of the system is described by a stochastic process and the observations made on the system are described by a stochastic process which is a function of the state stochastic process and noise. The control problem is to control the state of the system optimally (given a performance criterion) using the observations. These stochastic optimal control problems form a large class of physically important problems. The conditional probability of the state given all the past observations, which is obtained for the filtering problem, is the fundamental tool for determining the optimal control to be used because the conditional probability represents our probabilistic knowledge of the state of the system. The filtering solution with the conditional probability represents a major step to solving the stochastic optimal control problem.

Determining whether received data contain a signal and noise or merely noise has many applications particularly in radar problems where a signal is sent and then the received data are checked to determine whether the data contain a reflected signal and noise or only noise. To make the decision in an optimal manner between the two hypotheses that the data contain signal and noise or that the data contain noise we apply some results from statistical decision theory and calculate a likelihood function. This likelihood function determined from the data is then compared with a threshold to indicate the hypothesis to choose. For applications it is useful to be able to calculate this likelihood function recursively, i.e., to obtain a differential equation for the evolution in time of the likelihood function. This recursive form for the likeli-

hood function can often be obtained by applying some results from filtering theory.

Before analyzing the filtering and the detection problems in depth we shall discuss some of the history of these problems indicating the results that have been obtained, describe the results that we shall obtain and describe some of the mathematical techniques and results that will be used in analyzing the filtering and the detection problems.

A. DESCRIPTION AND HISTORY OF THE PROBLEMS

1. Nonlinear Filtering

The filtering problem of estimating one stochastic process given observations of a related stochastic process has received attention in both engineering and mathematics. Kalman and Bucy [Ref. 1] modeling the stochastic processes by linear differential equations with white noise inputs obtained a simple recursive solution to the linear filtering problem. The obvious extension of their work to a filtering problem with nonlinear differential equations (i.e., the nonlinear filtering problem) has been discussed by a number of authors, Stratonovich [Ref. 2], Kashyap [Ref. 3], Kushner [Refs, 4,5,6], Bucy [Ref. 7], and Mortensen [Ref. 8]. The original studies on this topic were somewhat naive and i⁺ was some time before it was realized that incorrect (or at least ambiguous) results had been obtained by not paying proper attention to some of the mathematical techniques involved. In particular, care had to be exercised in interpreting and manipulating certain integrals—the so-called Itô and Stratonovich stochastic integrals [Ref. 9].

The aim of the papers on this problem has been to derive a differential equation for the conditional probability density (or conditional

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moments) of the stochastic process to be estimated given all the past observations of a related stochastic process. The most general results for conditional moments that have been rigorously derived have been obtained by Kushner [Ref. 6]. He had to make several assumptions on the stochastic processes involved. Often these assumptions are difficult to verify for physical models. The physical meaning of many of the assumptions is unclear and often the assumptions were made only to obtain some mathematical results.

One reason for these many assumptions is that the problem is first solved in the discrete time and then there is a passage to the limit to obtain the continuous time result. Mortensen was the first to use a purely continuous time approach though he made some fairly restrictive assumptions.

2. Absolute Continuity of Measures

For the continuous time proof of the existence of the conditional probability density function we use certain results on the absolute continuity of probability measures that correspond to solutions of stochastic differential equations (stochastic differential equations will be defined subsequently). Prohorov [Ref. 10] obtained the first results for absolute continuity with the stochastic processes described by stochastic differential equations though some pioneering work on this problem was done by Cameron and Martin [Ref. 11]. Following Prohorov, Skorokhod [Refs. 12,13] and Girsanov [Ref. 14] obtained more general results on sufficient conditions for absolute continuity.

3. Detection Theory

Some detection theory problems of a stochastic signal in white

noise have been solved where the likelihood function (Radon-Nikodym derivative) can be recursively calculated. This recursive result was obtained by Schweppe [Ref. 15] for the case where the signal is generated by white noise into a finite dimensional linear system. His solution makes use of the linear filtering results of Kalman and Bucy [Ref. 1]. Van Trees [Ref. 16] has considered a related problem obtaining the same "type" of result as Schweppe. Sosulin and Stratonovich [Ref. 17] consider the signal as a general diffusion process and indicate that the nonlinear filtering results can be used to solve recursively for the likelihood function.

B. NEW RESULTS

We will briefly describe some of the results obtained in this dissertation.

1. Nonlinear Filtering Theory

We present a rigorous derivation of a stochastic equation for the evolution of the conditional probability density. The proof works directly with the continuous time stochastic processes and no "discretizations" are used. We also prove existence and differentiability properties for transition densities corresponding to diffusion solutions of stochastic differential equations. These properties are used in the derivation of the equation for the conditional probability density. The main results are Theorems 2.1 and 4.1.

2. Absolute Continuity of Measures

We derive necessary and sufficient conditions for the absolute continuity of measures corresponding to the solutions of a large class of stochastic differential equations. This result is given in Theorem 3.1.

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3. Detection Theory

We consider the detection problem of determining whether a stochastic signal (diffusion process) is present in white Gaussian noise (Brownian motion) i.e., we have the two hypotheses to test

$$dy_t = H(t)x_t \, dt + d\tilde{B}_t \quad \text{for} \quad \theta = 1$$

$$= d\tilde{B}_t \quad \text{for} \quad \theta = 0$$

where x_t is the signal and $d\tilde{B}_t$ is the noise. We rigorously derive a differential equation for the likelihood function and relate this to the nonlinear filtering problem. We compare this result to the results of Schweppe [Ref. 15] and Sosulin and Stratonovich [Ref. 17] and relate the differences to the different definitions of stochastic integral.

We consider the detection problem of a stochastic signal in correlated noise and discuss conditions for nonsingular detection. We show how the nonsingular problem can be related to a nonlinear filtering problem to obtain a differential equation for the likelihood function.

C. SOME MATHEMATICAL TECHNIQUES AND RESULTS

1. General Theory and Notation

A number of mathematical definitions and results will be used in this dissertation that may be somewhat unfamiliar to most engineers. We will briefly review these topics here.

Stochastic processes which are solutions of stochastic differential equations will be considered here. For general references on stochastic processes and particularly to stochastic differential equations the reader is referred to Doob [Ref. 18] and K. Itô [Ref. 19]. Some familiarity with

the basic definitions of probability theory and stochastic processes will be assumed. Generally a stochastic process could be denoted by the four-tuple $(\Omega, \mathfrak{F}, P, \{X_t\}_{t \in T})$ where

i) $(\Omega, \mathfrak{F}, P)$ is a probability space, i.e., a measurable space with a probability measure on it. For our case we will usually consider Ω to be the space of continuous functions on $T = [0,1]$, $\mathfrak{F}$ then is the Borel σ-algebra for Ω and the probability measure P is a measure on the space of continuous functions. The points in Ω will be denoted by ω.

ii) $\{X_t\}_{t \in T}$ is a family of random variables on $(\Omega, \mathfrak{F})$ with values in the state space $(E, \mathcal{E})$. For our case the state space $(E, \mathcal{E})$ will usually be $(\mathcal{R}^n, B^n)$ where B^n is the Borel σ-algebra on $\mathcal{R}^n$ (Euclidean n-space). The time set T will be a compact interval, usually $[0,1]$.

We define $B(X_u, u \leq t)$ as the Borel σ-algebra generated by the process $\{X_u, u \leq t\}$. A family of (sub) σ-algebras $\mathfrak{F}_t$ is said to be increasing if for $s \leq t$ $\mathfrak{F}_s \subset \mathfrak{F}_t$. The process $\{X_t\}$ is said to be adapted to $\mathfrak{F}_t$ if X_t is $\mathfrak{F}_t$ measurable. For example, X_t is adapted to $B(X_u, u \leq t)$. We will assume that all the (sub) σ-algebras are augmented, i.e., if $\mathfrak{n} = \{\Lambda : P(\Lambda) = 0\}$ then $\mathfrak{F}_t \supset \mathfrak{n}$ for $\forall t$. Without this assumption when we obtain almost sure (a.s.) equality we are not certain that all versions have the desired measurability properties on the sub σ-algebra.

By a Markov process we mean fundamentally a stochastic process that has the so-called Markov property (cf. Loève [Ref. 20]) i.e.,

$$P\left(\Lambda_{future} \middle| B_{present+past}\right) = P\left(\Lambda_{future} \middle| B_{present}\right) \quad a.s.$$

It will be useful to define more precisely the motion of a Markov process (cf. Dynkin [Refs. 21,22]). Take a measurable space $(E,\mathcal{E})$. The function $P(s,x;t,\Gamma)$ $(0 \le s \le t, x \in E, \Gamma \in \mathcal{E})$ is said to be a transition measure if the following conditions are satisfied:

a. $P(s,x;t,\Gamma)$ is a measure (as a function of the set Γ)

b. $P(s,x;t,\Gamma)$ is an $\mathcal{E}$ measurable function of x

c. $P(s,x;t,\Gamma) \le 1$

d. $P(s,x;s,E\backslash x) = 0$

e. $P(s,x;u,\Gamma) = \int_E P(s,x;t,dy)P(t,y;u,\Gamma) \quad 0 \le s \le t \le u$

We shall also need the notion of a transition density. Let μ be a measure on the state space $(E,\mathcal{E})$. The function $p(s,x;t,y)$ $(t > s; x,y \in E)$ is called a transition density if the following conditions are satisfied:

α. $p(s,x;t,y) \ge 0$ $(t > s; x,y \in E)$

β. For fixed t and s $p(s,x;t,y)$ is an $\mathcal{E} \times \mathcal{E}$ measurable function of (x,y)

γ. $\int_E p(s,x;t,y)\mu(dy) \le 1$ $(t > s, x \in E)$

δ. $p(s,x;t,y) = \int_E p(s,x;u,z)p(u,z;t,y)\mu(dz)$ $(s < u < t, x,y \in E)$

Under certain conditions on the Markov process it is possible to show that the transition density function exists and satisfies the following two linear second-order parabolic equations.

$$- \frac{\partial p(s,y;t,x)}{\partial s} = \sum_i a_i(s,y) \frac{\partial p}{\partial y_i} + \frac{1}{2} \sum_{i,j} c_{ij}(s,y) \frac{\partial^2 p}{\partial y_i \partial y_j} \qquad (1.1)$$

$$\frac{\partial p(s,y;t,x)}{\partial t} = - \sum_i \frac{\partial(a_i(t,x)p)}{\partial x_i} + \frac{1}{2} \sum_{i,j} \frac{\partial^2(c_{ij}(t,x)p)}{\partial x_i \partial x_j} \qquad (1.2)$$

where

$$\lim_{t \downarrow 0} \frac{1}{t} \int_{|y-x| < \delta} (y - x)P(s,x;s + t,dy) = a(s,x)$$

$$\lim_{t \downarrow 0} \frac{1}{t} \int_{|y-x| < \delta} (y - x)(y - x)^T P(s,x;s + t,dy) = c(s,x)$$

$$\delta > 0$$

These equations are usually called Kolmogorov's backward and forward equation respectively. The reader is referred to Feller [Ref. 23] and Bharcuha-Reid [Ref. 24] for a more complete discussion of these equations.

Consider a process $\{X_t\}$. If $\{X_t\}$ is a Markov process then for $t > \tau$ and $\Lambda \in B(X_t)$

$$P(\Lambda | X_u, u \leq \tau) = P(\Lambda | X_\tau) \quad \text{a.s.}$$

If we can replace τ by a (random) stopping time $T(\omega)$ such that

$$\{\omega : T \leq s\} \in B(X_u, u \leq s)$$

and if $T \leq \tau$ and

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$$P(\Lambda \,|\, X_u, \; u \leq T) \; = \; P(\Lambda \,|\, X_T) \quad \text{a.s.}$$

then $\{X_t\}$ is called a strong Markov process.

By a diffusion process we mean a strong Markov process with continuous sample paths.

By Brownian motion (also called the Wiener process) we mean a process $\{B(t,\omega),P\}$ which has continuous sample paths whose increments are independent and normally distributed. If $\{B_t\}$ is defined for $t \in [0,1]$ we assume $B(0,\omega) = 0$ and $E\left(B_t^2\right) = t$. By n-dimensional Brownian motion we mean a system of n one-dimensional Brownian motions independent of each other.

We now consider integrals with respect to the Brownian motion integrator, i.e., integrals of the form

$$\int f(t) \; dB(t,\omega)$$

Since Brownian motion has unbounded variation we cannot interpret this integral (for almost all ω) as a Lebesgue-Stieltjes integral. Wiener [Ref. 25] defined this integral using the integration theory developed by Daniell. This integral can be defined for all functions, f, that are square integrable (cf. Doob [Ref. 18] for a good discussion).

K. Itô [Refs. 19,26] considered the problem where f was a random function independent of the future Brownian motion. He first defined the integral for step functions as

$$\sum_i f(t_i,\omega)\left(B_{t_{i+1}} - B_{t_i}\right)$$

$\exists\ k > 0$ such that for $\forall\ x \in \mathcal{R}^n$

$$x^T A x > k x^T x = k|x|^2 \qquad (1.6)$$

The proof of a theorem, lemma, etc., begins with the word "proof" and terminates with the symbol ∎ which can be read as "this completes the proof."

2. Theory of Stochastic Differential Equations

We will have occasion throughout this dissertation to consider vector stochastic differential equations such as

$$dx(t,\omega) = a(t,x(t,\omega))\ dt + b(t,x(t,\omega))\ dB(t,\omega) \qquad (1.7)$$

where $x(t,\omega)$, $a(t,x(t,\omega))$ and $B(t,\omega)$ will be $n \times 1$ column vectors and b will be an $n \times n$ matrix. The process $\{B_t\}$ is n-dimensional Brownian motion. The vector a is usually referred to as the drift or transfer vector and the matrix b is called the diffusion matrix.

We shall briefly review some results from the theory of stochastic differential equations that will be used in later chapters.

a. Existence and Uniqueness of Solutions of Stochastic Differential Equations

The usual results for existence and uniqueness for solutions of stochastic differential equations are due to K. Itô [Refs. 19,26] and I. I. Gikhman [Ref. 27].

Theorem 1.1. Consider a vector stochastic differential equation

$$dx(t,\omega) = a(t,x(t,\omega))\ dt + b(t,x(t,\omega))\ dB(t,\omega) \qquad (1.7)$$

and showed that this definition could be extended to all functions f satisfying

$$\int_\Omega \int_T |f(t,\omega)|^2 \, dt \, dP < \infty$$

If the integrand is measurable with respect to the past Brownian motion then by Itô's definition of stochastic integrals this integral with respect to Brownian motion is a martingale of Brownian motion, i.e., for $\tau < t$

$$E\left[\int_0^t f(s,\omega) \, dB_s \Big| B(B_u, u \leq \tau)\right] = \int_0^\tau f(s,\omega) \, dB_s \quad \text{a.s.} \qquad (1.3)$$

This martingale property will be important in many of our calculations.

Some other notational descriptions will be useful. By $\sigma(L^1, L^\infty)$ we mean the weak topology induced on L^1 by L^∞. A description of weak topology can be found in Royden [Ref. 27] or Kelley [Ref. 28].

Given a matrix $a(t,x) = (a_{ij}(t,x))$ we say that $a(t,x)$ satisfies a global Lipschitz condition if each component a_{ij} satisfies this property, i.e., $\forall x, y$

$$|a_{ij}(t,x) - a_{ij}(t,y)| \leq K|x - y| \qquad i,j = 1,2,\ldots,n \qquad (1.4)$$

Similarly by $a(t,x)$ being bounded we mean $\exists K < \infty$ such that for $\forall t, x$

$$|a_{ij}(t,x)| \leq K \qquad i,j = 1,2,\ldots,n \qquad (1.5)$$

Given a vector or a matrix A we denote the transpose of A as A^T. By a symmetric matrix A being strictly positive definite we mean

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where $t \in [s,1]$, $x(s,\omega) = \alpha(\omega)$, $P\{|\alpha| < \infty\} = 1$ and the terms of the vector $a(t,x)$ and the matrix $b(t,x)$ satisfy a global Lipschitz condition in x and are measurable in t. Then the solution $\{x_t\}$ exists, is unique, and is a diffusion process. Furthermore if $\alpha \in L^2$ then

$$\sup_{s \leq t \leq 1} E\left(x_t^2\right) < \infty$$

Proof.

The idea of the proof is to use Picard iteration, as

$$x^o(t,\omega) = \alpha$$

$$x^{n+1}(t,\omega) = \alpha + \int_s^t a(u,x^n(u,\omega))\,du + \int_s^t b(u,x^n(u,\omega))\,dB(u,\omega) \quad (1.8)$$

to show that a solution exists and that it is unique. The integral with respect to Brownian motion is the stochastic integral defined by K. Itô. For details of the proof and the stochastic integral the reader is referred to K. Itô [Refs. 19,26]. ∎

b. Stochastic Differential Rule

Another result from stochastic differential equation theory will be important in the following presentation, that is, the stochastic differential rule. It is known that twice continuously differentiable functions of diffusion processes violate some of the usual rules for transformations in ordinary calculus. The stochastic differential rule is described in the following theorem which is due to K. Itô [Ref. 30].

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<u>Theorem 1.2.</u> Let $x(t,\omega)$ satisfy

$$dx(t,\omega) \;=\; a(t,\omega)\,dt + b(t,\omega)\,dB(t,\omega) \tag{1.9}$$

where we assume the vector $a(t,\omega)$ and the matrix $b(t,\omega)$ are independent of the future Brownian motion and G is an open subset of the n-space $\mathfrak{R}^n$ which contains all the points $(x(t,\omega))$ $u \leq t \leq v$ $\omega \in \Omega$. Let $f(t,x)$ be a continuous function defined for $u \leq t \leq v$ $x = (x_1, x_2, \ldots, x_n)^T \in G$ and suppose that

$$f^0(t,x) \;=\; \frac{\partial f(t,x)}{\partial t} \tag{1.10}$$

$$f^i(t,x) \;=\; \frac{\partial f(t,x)}{\partial x_i} \qquad i = 1,2,\ldots,n \tag{1.11}$$

$$f^{ij}(t,x) \;=\; \frac{\partial^2 f(t,x)}{\partial x_i \partial x_j} \qquad i,j = 1,2,\ldots,n \tag{1.12}$$

are all continuous. Then the differential of $\eta(t,\omega) \equiv f(t,x(t,\omega))$ is

$$
\begin{aligned}
d\eta(t,\omega) \;=\; &\left(\left(f^0(t,x(t,\omega)) + \sum_i f^i(t,x(t,\omega))\right)a_i(t,\omega)\right. \\
&\left. + \frac{1}{2}\sum_{i,j} f^{ij}(t,x(t,\omega))c_{ij}(t,\omega)\right) dt \\
&+ \sum_{i,j} f^i(t,x(t,\omega))b_{ij}(t,\omega)\,dB_j(t,\omega) \tag{1.13}
\end{aligned}
$$

where $c(t,\omega) = \{c_{ij}(t,\omega)\} = b^T(t,\omega)b(t,\omega)$.

Proof.

We briefly sketch the proof to give the reader an idea of the techniques. By the Taylor expansion of $f(t, x^1, \ldots, x^n)$ we have

$$\eta(s, \omega) - \eta(t, \omega) = \sum_{k=1}^{m} \left(\eta\left(t_k^m, \omega\right) - \eta\left(t_{k-1}^m, \omega\right) \right)$$

$$= \sum_{k=1}^{m} \left[f^0(\gamma)\left(t_k^m - t_{k-1}^m\right) + \sum_{i=1}^{n} f^i(\gamma)\left(x_i\left(t_k^m\right) - x_i\left(t_{k-1}^m\right)\right) \right.$$

$$+ \frac{1}{2} \sum_{i,j} f^{ij}(\gamma)\left(x_i\left(t_k^m\right) - x_i\left(t_{k-1}^m\right)\right)\left(x_j\left(t_k^m\right) - x_j\left(t_{k-1}^m\right)\right)$$

$$\left. + \sum_{i,j} \theta_{ijk}^m\left(x_i\left(t_k^m\right) - x_i\left(t_{k-1}^m\right)\right)\left(x_j\left(t_k^m\right) - x_j\left(t_{k-1}^m\right)\right) \right]$$

$$(1.14)$$

where $\gamma = \left(t_{k-1}^m, x_1\left(t_{k-1}^m\right), \ldots, x_n\left(t_{k-1}^m\right)\right)$, $t_k^m = t + (k/m)(s - t)$. Since $f^{ij}(t, x_1, \ldots, x_n)$ are continuous and $x_i(t, \omega)$ $i = 1, 2, \ldots, n$ are all continuous in t a.s., θ_{ijk}^m tends to 0 uniformly in m and k as $n \to \infty$ a.s. Therefore the last term in the above expression goes to zero in probability. It can be shown that

$$(x_i(s, \omega) - x_i(t, \omega))(x_j(s, \omega) - x_j(t, \omega))$$

$$= \int_t^s \left[(x_i(\tau, \omega) - x_i(t, \omega))a_j(\tau, \omega) + (x_j(\tau, \omega) - x_j(t, \omega))a_i(\tau, \omega) \right.$$

$$\left. + c_{ij}(\tau, \omega) \right] d\tau$$

$$+ \int_t^s \left[(x_i(\tau, \omega) - x_i(t, \omega))b_{ik}(\tau, \omega) + (x_j(\tau, \omega) - x_j(t, \omega))b_{ik}(\tau, \omega) \right] dB_k(\tau, \omega)$$

$$(1.15)$$

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Using this result the remaining terms in the Taylor expansion are:
(suppressing the summation signs)

$$
\int_t^s \left(f^0(\alpha) + f^i(\alpha)a_i(\tau) + \frac{1}{2} f^{ij}(\alpha)c_{ij}(\tau) \right) d\tau + \int_t^s f^i(\alpha)b_{ij}(\tau) \, dB_j(\tau)
$$

$$
+ \frac{1}{2} \int_t^s f^{ij}(\alpha) \left[(x_i(\tau) - x_i(\lambda_m(\tau)))a_j(\tau) + (x_j(\tau) - x_j(\lambda_m(\tau)))a_i(\tau) \right] d\tau
$$

$$
+ \frac{1}{2} \int_t^s f^{ij}(\alpha) \left[(x_i(\tau) - x_i(\lambda_m(\tau)))b_{jk}(\tau) + (x_j(\tau) - x_j(\lambda_m(\tau)))b_{ik}(\tau) \right] dB_k(\tau)
$$

where $\lambda_m(\tau)$ denotes the maximum t_k^m which does not exceed τ and $\alpha = (\lambda_m(\tau), x_1(\lambda_m(\tau)), \ldots, x_n(\lambda_m(\tau)))$. Since $x_i(\lambda_m(\tau)) \to x_i(\tau)$ a.s. the last two integrals in the above expression go to zero in the limit and we have the result. ∎

To illustrate the application of the stochastic differential rule we provide two examples (which will also be used subsequently).

Example 1.

Consider the function M_t given by

$$
M_t = \exp \left[\int_s^t a^T(u, x(u, \omega)) c^{-1}(u, x(u, \omega)) \, dx(u, \omega) \right.
$$

$$
\left. - \frac{1}{2} \int_s^t a^T(u, x(u, \omega)) c^{-1}(u, x(u, \omega)) a(u, x(u, \omega)) \, du \right] \qquad (1.16)
$$

where $dx(u, \omega) = b(u, x(u, \omega)) \, dB(u, \omega)$, $c = b^T b$, and b^{-1} exists.

The function M_t can therefore be rewritten as

$$M_t = \exp\left[\int_s^t a^T(u,x(u,\omega))b^{T^{-1}}(u,x(u,\omega))\ dB(u,\omega)\right.$$

$$\left. - \frac{1}{2}\int_s^t a^T(u,x(u,\omega))c^{-1}(u,x(u,\omega))a(u,x(u,\omega))\ du\right] \qquad (1.17)$$

Let

$$dz_t = a^T(t,x_t)b^{T^{-1}}(t,x_t)\ dB_t$$

$$- \frac{1}{2}\ a^T(t,x_t)c^{-1}(t,x_t)a(t,x_t)\ dt \qquad (1.18)$$

Then

$$M_t = e^{z_t} \qquad (1.19)$$

We shall now apply the stochastic differential rule (Theorem 1.2) to the function e^{z_t}. We first compute the derivatives of e^x

$$e^x = \frac{d}{dx}\ e^x = \frac{d^2 e^x}{dx^2}$$

Substituting these terms in Eq. (1.13) we have the following equation for M_t

$$M_t = 1 + \int_s^t M_u a^T(u,x_u)b^{T^{-1}}(u,x_u)\ dB_u$$

$$- \frac{1}{2}\int_s^t M_u a^T(u,x_u)c^{-1}(u,x_u)a(u,x_u)\ du$$

$$+ \frac{1}{2}\int_s^t M_u a^T(u,x_u)b^{T^{-1}}(u,x_u)b^{-1}(u,x_u)a(u,x_u)\ du \qquad (1.20)$$

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Recalling that $c = b^T b$ we have

$$M_t = 1 + \int_s^t M_u a^T(u,x_u) b^{T^{-1}}(u,x_u)\, dB_u \qquad (1.21)$$

or written as only a functional of x_u, $s \le u \le t$ we have

$$M_t = 1 + \int_s^t M_u a^T(u,x_u) c^{-1}(u,x_u)\, dx_u \qquad (1.22)$$

Example 2.

Find the differential for q_t^{-1} where

$$q_t = E_X \exp\left[\int_s^t g^T(u,x_u,y_u)\, dB_u - \frac{1}{2} \int_s^t g^T(u,x_u,y_u) g(u,x_u,y_u)\, du\right]$$

$$= 1 + E_X \int_s^t \psi_u g(u,x_u,y_u)\, dB_u \qquad (1.23)$$

where $q_t = E_X(\psi_t)$ and we assume

$$q_t = 1 + \int_s^t E_X(\psi_u g(u,x_u,y_u))\, dB_u \qquad (1.24)$$

The expression for the differential of q_t^{-1} can be written down formally as

$$d\left(q_t^{-1}\right) = -\frac{1}{q_t^2}\, dq_t + \frac{1}{2}\frac{2}{q_t^3} (dq_t)^2 \qquad (1.25)$$

since the differential rule can be characterized as

$$df_t = f_t'\, dx_t + \frac{1}{2} f_t''(dx_t)^2 \qquad (1.26)$$

and since the term $(dq_t)^2$ arises only from the stochastic integral via

our differential rule we have

$$d\left(q_t^{-1}\right) = -\frac{1}{q_t^2} E_X(\psi_t g(t,x_t,y_t)) \, dB_t + \frac{1}{q_t^3} E_X(\psi_t g(t,x_t,y_t))^T E_X(\psi_t g(t,x_t,y_t)) \, dt$$

(1.27)

c. Sufficient Conditions for the Absolute Continuity
of Diffusion Processes

We now consider the following stochastic differential

equations.

$$dx(t,\omega) = a(t,x(t,\omega)) \, dt + b(t,x(t,\omega)) \, dB(t,\omega)$$ (1.7)

$$dy(t,\omega) = f(t,y(t,\omega)) \, dt + g(t,y(t,\omega)) \, dB(t,\omega)$$ (1.28)

Almost all sample functions of these two stochastic processes $\{x_t\}$ and
$\{y_t\}$ are continuous functions. Therefore we can describe the stochastic
processes $\{x_t\}$ and $\{y_t\}$ by measures, say μ_X and μ_Y on the
n-dimensional space of continuous functions $C_n[s,1]$. We shall give
sufficient conditions for μ_Y to be absolutely continuous with respect
to μ_X (written $\mu_Y \ll \mu_X$).

In terms of stochastic differential equations the first results were
obtained by Prohorov [Ref. 10] though some important pioneering work in
Wiener measure (the measure induced by Brownian motion) was done by
Cameron and Martin [Ref. 11]. Subsequent to Prohorov, Skorokhod [Ref. 13]
and Girsanov [Ref. 14] considered the problem and obtained more general
results. We state the result due to Girsanov in the following theorem.

Theorem 1.3. Suppose that

$$lx(t,\omega) = a(t,x(t,\omega)) \, dt + b(t,x(t,\omega)) \, dB(t,\omega) \qquad (1.7)$$

$$dy(t,\omega) = (a(t,y(t,\omega)) + b(t,y(t,\omega))h(t,y(t,\omega))) \, dt + b(t,y(t,\omega)) \, dB(t,\omega)$$
$$\qquad (1.29)$$

where

    i) $t \in [s,1]$

    ii) $h(t,y(t,\omega)) = (h_1(t,y(t,\omega)), h_2(t,y(t,\omega)), \ldots, h_n(t,y(t,\omega)))^T$

    iii) $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are measurable in both variables

    iv) $\displaystyle\int_s^1 |b(t,x(t,\omega))|^2 \, dt < \infty \quad$ a.e.

$$\int_s^1 |h(t,x(t,\omega))|^2 \, dt < \infty \quad \text{a.e.}$$

$$\int_s^1 |a(t,x(t,\omega))|^2 \, dt < \infty \quad \text{a.e.}$$

    v) $|h(t,x(t,\omega))| < h_0(|x(t,\omega)|)$

where h_0 is a nondecreasing function of a real variable. Then

$$\mu_Y \ll \mu_X$$

where μ_X and μ_Y are the measures induced on $C_n[s,1]$ by $\{x_t\}$ and $\{y_t\}$ respectively.

The Radon-Nikodym derivative, $d\mu_Y/d\mu_X$, will be given by

$$\frac{d\mu_Y}{d\mu_X} = \exp\left[\int_s^1 h^T(u,x_u) \, dB_u - \frac{1}{2}\int_s^1 |h(u,x_u)|^2 \, du\right] \qquad (1.30)$$

If b^{-1} exists, this can be rewritten entirely in terms of $\{x_t\}$,

$$\frac{d\mu_Y}{d\mu_X} = \varphi_1(x_u, \, s \leq u \leq 1) = \exp\left[\int_s^1 h^T(u,x_u)b^{-1}(u,x_u) \, dx_u\right.$$

$$\left. - \int_s^1 h^T(u,x_u)b^{-1}(u,x_u)a(u,x_u) \, du - \frac{1}{2}\int_s^1 |h(u,x_u)|^2 \, du\right]$$

$$(1.31)$$

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II.   TRANSITION PROBABILITY DENSITIES FOR DIFFUSION PROCESSES

Markov processes which are solutions of stochastic differential equations generated by Brownian motion are often used to describe the nonlinear filtering problem and the stochastic optimal control problem. A fundamental tool for these problems is the conditional density, viz., the probability density for the process to be estimated conditioned on all the past observations. The expression for the conditional probability density is a function of the transition density for the Markov process which is to be estimated. Therefore to derive an expression for the conditional probability density rigorously it is first necessary to prove that this associated transition density exists. To derive a stochastic equation for the conditional probability density it is necessary to prove that the transition density is differentiable enough to satisfy Kolmogorov's forward equation (the Fokker-Planck equation).

In this chapter we shall consider stochastic differential equations which have diffusion process solutions, and (i) prove that the transition density with respect to Lebesgue measure exists for the diffusion process, (ii) prove that this transition density is suitably differentiable and that the various derivatives can be bounded so that the density function can be characterized as the fundamental solution of Kolmogorov's equations.

A.   EXISTENCE AND DIFFERENTIABILITY OF TRANSITION DENSITIES

We now consider the problem of showing that the solution of the following vector stochastic differential equation

$$dx(t,\omega) = a(t,x(t,\omega))\, dt + b(t,x(t,\omega))\, dB(t,\omega) \qquad (2.1)$$

with suitable assumptions on the coefficients has a transition density
and that this density satisfies Kolmogorov's forward and backward equations.
In previous work on this problem, both a and b were assumed to be
bounded and Hölder (or Lipschitz) continuous. Under these assumptions,
Mortensen [Ref. 8] established existence of the density, while Dynkin
[Ref. 21] proved that the density existed and that it satisfied Kolmogorov's
equations.

We make the following assumptions on the coefficients

    i)   The diffusion matrix $b(t,x)$ is Hölder continuous in

        t, globally Lipschitz continuous in x and globally bounded.

        Moreover, the symmetric matrix $c \left(c = b^T b \right)$ is strictly

        positive definite. The terms

$$\frac{\partial c_{ij}(t,x)}{\partial x_i} \, , \, \frac{\partial^2 c_{ij}(t,x)}{\partial x_i \partial x_j} \qquad i,j = 1,2,\ldots,n$$

        are globally Lipschitz continuous in x, continuous in t

        and globally bounded.

   ii)   The transfer (drift) vector $a(t,x)$ is continuous in t and

        globally Lipschitz continuous in x. The terms

$$\frac{\partial a_i(t,x)}{\partial x_i} \qquad i = 1,2,\ldots,n$$

        are globally Lipschitz continuous in x and continuous in

        t.

We state our result in the following theorem.

<u>Theorem 2.1.</u> Let $x(t,\omega)$ satisfy

$$dx(t,\omega) = a(t,x(t,\omega))\, dt + b(t,x(t,\omega))\, dB(t,\omega) \qquad (2.1)$$

where we make the assumptions on the coefficients described above.
Then there exists a version of the transition density for $\{x_t\}$, p_X,
which satisfies Kolmogorov's equations.

Before presenting the proof we shall briefly outline the steps. We
first show that Kolmogorov's backward equation is naturally associated
with the stochastic differential equation describing $\{x_t\}$ (Lemma 2.1).
If this backward equation has a unique fundamental solution then we can
show that this fundamental solution is the transition probability density
for $\{x_t\}$ (Lemma 2.2). Furthermore, if we can show that the formal
adjoint of the backward equation has a unique fundamental solution and
that for large values of the space coordinates the fundamental solution
decreases sufficiently rapidly, then we can prove that the transition
density satisfies Kolmogorov's forward equation.

Since the coefficient $a(t,x_t)$ in the stochastic differential equation
can be unbounded the usual results for existence and uniqueness of funda-
mental solutions for linear second-order parabolic equations cannot be
used. We proceed by first showing existence of the transition density
relating it to a simpler process (Lemma 2.3) and then finally proving that
the transition density is suitably differentiable (Lemma 2.4).

1. <u>Kolmogorov's Equations</u>

Since we want to show that a transition density for a diffusion
process exists and satisfies Kolmogorov's equations we have to use some

techniques and results from the theory of partial differential equations.

In particular it will be useful to define a fundamental solution of a partial differential equation.

<u>Definition.</u> A function $p(s,y;t,x)$ defined for $x,y \in \mathcal{R}^n$ and $s < t$, $s,t \in T = [0,1]$ is a fundamental solution of $\mathcal{L}f = 0$ if it has the following two properties

a. Considered as a function of (x,t) for each fixed $(s,y) \in \mathcal{R}^n \otimes [0,1]$ the derivatives of p which appear in $\mathcal{L}$ exist, are continuous, and satisfy

$$\mathcal{L}p = 0 \text{ in } \mathcal{R}^n \otimes (0,1) \qquad (2.2)$$

b. If h is a continuous real valued function on $\mathcal{R}^n$ with compact support then

$$\lim_{(x,t) \to (x_0, s+)} \int_{\mathcal{R}^n} p(s,y;t,x)h(y)\, dy = h(x_0) \qquad (2.3)$$

We will now associate Kolmogorov's equations with the vector stochastic differential equation

$$dx(t,\omega) = a(t,x(t,\omega))\, dt + b(t,x(t,\omega))\, dB(t,\omega) \qquad (2.1)$$

<u>Lemma 2.1.</u> Let g be a bounded real valued twice continuously differentiable function defined on $\mathcal{R}^n$. Let $\{x_t\}$ satisfy

$$dx(t,\omega) = a(t,x(t,\omega))\, dt + b(t,x(t,\omega))\, dB(t,\omega) \qquad (2.1)$$

where we assume the vector $a(t,x)$ and the matrix $b(t,x)$ satisfy a global Lipschitz condition in x and are continuous in t, and the matrix c $\left(c = b^T b\right)$ is strictly positive definite and $x(s) = y$.

Let

$$f(s,y;t) = E_{(s,y)} g(x(t,\omega)) \qquad (2.4)$$

Then $f(s,y;t)$ as a function of (s,y) satisfies the following linear second-order parabolic equation

$$-\frac{\partial f}{\partial s} = \sum_i a_i(s,y) \frac{\partial f}{\partial y_i} + \frac{1}{2} \sum_{i,j} c_{ij}(s,y) \frac{\partial^2 f}{\partial y_i \partial y_j} \qquad (2.5)$$

Proof.

This result follows from the stochastic differential rule (Dynkin [Ref. 21]) and Dynkin's formula. ∎

We now show that if the partial differential equation described in the above lemma has a unique fundamental solution, then by the properties of a fundamental solution we can characterize it as the transition density of $\{x_t\}$.

Lemma 2.2. If the linear parabolic equation in Lemma 2.1

$$-\frac{\partial f}{\partial s} = \sum_i a_i(s,y) \frac{\partial f}{\partial y_i} + \frac{1}{2} \sum_{i,j} c_{ij}(s,y) \frac{\partial^2 f}{\partial y_i \partial y_j} \qquad (2.5)$$

has a unique fundamental solution, p, then p is the transition density for the diffusion process $\{x_t\}$ which satisfies

$$dx(t,\omega) = a(t,x(t,\omega))\,dt + b(t,x(t,\omega))\,dB(t,\omega)$$

and therefore the transition density, p, satisfies Kolmogorov's backward equation.

Proof.

Using the definition of fundamental solution we can show that $f(s,y;t)$ can be expressed as

$$f(s,y;t) = \int g(x)p(s,y;t,x)\,dx \qquad (2.6)$$

We recall a standard result from measure theory that bounded twice continuously differentiable functions can approximate in measure any essentially bounded function, i.e., bounded twice continuously differentiable functions are dense in the weak topology $\sigma\left(L^1,L^\infty\right)$ induced on the space L^1 by the space L^∞ (cf. Royden [Ref. 27] or Halmos [Ref. 31]). Therefore the probability for any Borel set can be obtained from the fundamental solution as a limit of twice continuously differentiable functions. The other properties of a transition density also follow from the properties of a fundamental solution. ∎

We have therefore shown that the transition density satisfies Kolmogorov's backward equation.

We will now sketch the arguments to obtain the forward equation. The formal adjoint of the backward equation is

$$\frac{\partial f}{\partial t} = -\sum_i \frac{\partial(a_i(t,x)f)}{\partial x_i} + \frac{1}{2}\sum_{i,j} \frac{\partial^2(c_{ij}(t,x)f)}{\partial x_i \partial x_j} \qquad (2.7)$$

Assuming that a fundamental solution, p, exists and is unique for the backward equation, to show that p satisfies the adjoint equation (Kolmogorov's forward equation), besides assuming the appropriate differentiability of a, c and p, we must assume that the following terms, obtained by integrating by parts to derive the forward equation, are zero.

$$p(s,y;t,x)a_i(t,x) \Big|_{-\infty}^{\infty} = 0 \qquad i = 1,2,\ldots,n \qquad (2.8)$$

$$p(s,y;t,x)c_{ij}(t,x) \Big|_{-\infty}^{\infty} = 0 \qquad i,j = 1,2,\ldots,n \qquad (2.9)$$

$$\frac{\partial(p(s,y;t,x)c_{ij}(t,x))}{\partial x_i} \Big|_{-\infty}^{\infty} = 0 \qquad i,j = 1,2,\ldots,n \qquad (2.10)$$

We have shown that if Kolmogorov's backward equation has a unique fundamental solution then this fundamental solution is the transition density for the process $\{x_t\}$. If we assume the appropriate differentiability assumptions for the coefficients a and c to make Kolmogorov's forward equation meaningful and we assume this forward equation has a unique fundamental solution with the above equations (Eqs. 2.8, 2.9, 2.10) being satisfied then we will have shown that the transition density satisfies Kolmogorov's forward equation.

2. Usual Results for Existence and Uniqueness of Fundamental Solutions of Linear Parabolic Equations

To prove existence and uniqueness of the fundamental solutions of Kolmogorov's equations we make use of some of the usual results for fundamental solutions of linear second-order parabolic equations. We review these results now.

Consider the general linear homogeneous second-order parabolic equation in the strip $H = (0,1] \otimes \mathfrak{R}^n$

$$\mathcal{L}_{s,y} u \equiv \sum_{i,j=1}^{n} a_{ij}(s,y)\, \frac{\partial^2 u}{\partial y_i \partial y_j} + \sum_{i=1}^{n} b_i(s,y)\, \frac{\partial u}{\partial y_i} - c(s,y)u + \frac{\partial u}{\partial s} = 0 \tag{2.11}$$

The following theorem is due to Il'in, Kalashnikov and Oleinik [Ref. 32] (cf. also Refs. 33,34).

Theorem 2.2. Suppose that all the coefficients of the above equation are bounded and continuous in $\overline{H}$ in the set of variables s,y and that they satisfy a Hölder condition in y:

$$\left.
\begin{aligned}
|a_{ij}(s,y') - a_{ij}(s,y)| &\leq M|y' - y|^{\lambda} \\[2mm]
|b_i(s,y') - b_i(s,y)| &\leq M|y' - y|^{\lambda} \\[2mm]
|c(s,y') - c(s,y)| &\leq M|y' - y|^{\lambda}
\end{aligned}
\right\}
\quad
\begin{aligned}
& i,j = 1,2,\ldots,n \\
& \lambda > 0
\end{aligned}$$

In addition suppose that the coefficients a_{ij} satisfy in H a Hölder condition in s:

$$|a_{ij}(s',y) - a_{ij}(s,y)| \leq M|s' - s|^{\lambda}$$

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and

$$\sum_{i,j=1}^{n} a_{ij}(s,y)\alpha_i\alpha_j \geq \mu \sum_{i=1}^{n} \alpha_i^2 \qquad \mu > 0$$

for all $(s,y) \in \overline{H}$ and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then the parabolic equation has a fundamental solution $p(s,y;t,x)$ and this solution is unique. For $p(s,y;t,x)$ we have the following estimates

$$p(s,y;t,x) > 0, \quad s < t, \quad s,t \in [0,1],$$

$$p(s,y;t,x) \leq K(t-s)^{-n/2} \exp\left[-\alpha|y-x|^2/(t-s)\right],$$

$$\left| \frac{\partial p(s,y;t,x)}{\partial y_i} \right| \leq K(t-s)^{-(n+1)/2} \exp\left[-\alpha|y-x|^2/(t-s)\right],$$

$$\left| \frac{\partial^2 p(s,y;t,x)}{\partial y_i \partial y_j} \right| \leq K(t-s)^{-n/2-1} \exp\left[-\alpha|y-x|^2/(t-s)\right],$$

where K and α are positive constants.

If the derivatives

$$\frac{\partial a_{ij}}{\partial y_i} , \quad \frac{\partial^2 a_{ij}}{\partial y_i \partial y_j} , \quad \frac{\partial b_i}{\partial y_i} \qquad i,j = 1,2,\ldots,n$$

are bounded and continuous in H and satisfy a Hölder condition in y, then $p(s,y;t,x)$ as a function of t and x satisfies the equation

$$\mathcal{L}_{t,x}^* p \equiv \sum_{i,j} \frac{\partial^2[a_{ij}(t,x)p]}{\partial x_i \partial x_j} - \sum_{i} \frac{\partial[b_i(t,x)p]}{\partial x_i} - c(t,x)p - \frac{\partial p}{\partial t} = 0$$

(2.12)

3. Existence of the Transition Density

To show the existence of the transition density for $\{x_t\}$ we apply a technique used by Mortensen [Ref. 8]. We first introduce a "simpler" process $\{y_t\}$ satisfying

$$dy(t,\omega) = b(t,y(t,\omega)) \, dB(t,\omega) \qquad (2.13)$$

Using the results for existence and uniqueness of the fundamental solutions for linear second-order parabolic equations with bounded Hölder continuous coefficients (Theorem 2.2) we can easily show that the transition density corresponding to $\{y_t\}$, p_Y, exists, is unique and satisfies Kolmogorov's forward and backward equations

$$-\frac{\partial p_Y(s,y;t,x)}{\partial s} = \frac{1}{2}\sum_{i,j} c_{ij}(s,y) \frac{\partial^2 p_Y}{\partial y_i \partial y_j} \qquad (2.14)$$

$$\frac{\partial p_Y(s,y;t,x)}{\partial t} = \frac{1}{2}\sum_{i,j} \frac{\partial^2 (c_{ij}(t,x)p_Y)}{\partial x_i \partial x_j} \qquad (2.15)$$

To show that the transition density for $\{x_t\}$, p_X, exists and is suitably differentiable is more difficult because a can be unbounded. We first obtain the existence of p_X by using the results for absolute continuity of the measures of solutions of stochastic differential equations (Theorem 1.3).

Lemma 2.3. Let $\{y_t\}$ and $\{x_t\}$ satisfy

$$dy(t,\omega) = b(t,y(t,\omega)) \, dB(t,\omega) \qquad (2.13)$$

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$$dx(t,\omega) \;=\; a(t,x(t,\omega))\,dt + b(t,x(t,\omega))\,dB(t,\omega) \qquad (2.1)$$

where $t \in [s,1]$ $x(s) = y(s) = y$.

Then $\mu_X \ll \mu_Y$ and the transition density exists for the process $\{x_t\}$ and a version of it for all $t \in [s,1]$ is given by

$$p_X(s,y;t,x) \;=\; E[\varphi_t \,|\, x_t = x]\, p_Y(s,y;t,x) \qquad (2.16)$$

where μ_X and μ_Y are the measures induced on $C_n[s,1]$ by $\{x_t\}$ and $\{y_t\}$ respectively and

$$\varphi_t(y_u,\; s \le u \le t) \;=\; \exp\left[\int_s^t a^T(u,y_u) c^{-1}(u,y_u)\, dy_u \right.$$

$$\left. - \frac{1}{2}\int_s^t |\, b^{-1}(u,y_u) a(u,y_u)\,|^2\, du \right] \qquad (2.17)$$

$c = b^T b$. The function φ_t is the Radon-Nikodym derivative $d\mu_X/d\mu_Y$ for the processes $\{x_u\}$ and $\{y_u\}$ $s \le u \le t$.

Proof.

The fact that $\mu_X \ll \mu_Y$ follows from Girsanov's theorem (Theorem 1.3). Fix t. Let $\Lambda \in B(x_t)$. Then

$$\mu_X(\Lambda) \;=\; \int_\Lambda \varphi_1 \, d\mu_Y \;=\; \int_\Lambda \varphi_t \, d\mu_Y \qquad (2.18)$$

We can replace φ_1 by φ_t in the above equation because $(\varphi_t,\, B(B_u,\; s \le u \le t))$ is a martingale. This will be discussed further

in the next chapter (Lemma 3.5).

$$\mu_X(\Lambda) \;=\; \int_\Lambda \varphi_t \; d\mu_Y \;=\; \int_\Lambda E[\varphi_t \,|\, x_t] \; d\mu_Y$$

$$=\; \int_\Lambda E[\varphi_t \,|\, x_t = x] p_Y(s,y;t,x) \; dx \qquad (2.19)$$

By the Radon-Nikodym theorem we have

$$p_X(s,y;t,x) \;=\; E[\varphi_t \,|\, x_t = x] p_Y(s,y;t,x) \quad \text{a.e.} \quad dx \qquad (2.20)$$

We can also immediately obtain this result for a countable set S dense in $T = (s,1]$. To show that this representation is valid for all $t \in (s,1]$ we proceed as follows. Let $t \notin S$, $t \in (s,1]$ and consider sets $\Lambda_a = \{x_t < a\} \in B(x_t)$, $a \in \mathscr{R}^n$. These probabilities can be obtained from S by the continuity of the sample paths and these sets generate $B(x_t)$. So we can obtain, via a limit, the conclusion for $\forall \; \Lambda \in B(x_t)$. ∎

4. Differentiability of the Transition Density

Lemma 2.4. The transition density for the process $\{x_t\}$, p_X, satisfies Kolmogorov's equations.

Proof.

We will consider primarily Kolmogorov's forward equation since the additional results for the backward equation follow by the same techniques. We recall a few preliminaries first. Let

$$\Omega_M = \left\{ \omega : \sup_{s \leq t \leq 1} |x(t,\omega)| < M \right\} \tag{2.21}$$

Since $\{x_t\}$ has continuous sample paths we have

$$P\left(\bigcup_M \Omega_M \right) = 1 = \lim_{M \to \infty} P(\Omega_M) \tag{2.22}$$

Also

$$\sup_{s \leq t \leq 1} E\left(x_t^2 \right) < \infty$$

Heuristically our approach is quite simple. Consider sets $\Gamma_n = \{x : |x| < n\} \otimes (0,1]$. We will give a Green's function for Γ_N which is not difficult to construct since the coefficients of the partial differential equation are bounded on this set. For large M, $P(\Omega_M) \approx 1$ so that a description of the transition density when the coefficients are bounded is (in some sense) a good approximation for the unbounded case. We will show that the sequence of Green's functions is monotone increasing. Since we showed that the density function for the diffusion process $\{x_t\}$ exists, we are able to bound this increasing sequence.

For Γ_n the Green's function $p_n(s,y;t,x)$ for

$$\mathcal{L}(\cdot) = \frac{\partial(\cdot)}{\partial t} - \frac{1}{2} \sum_{i,j} \frac{\partial^2 (c_{ij}(t,x)\cdot)}{\partial x_i \partial x_j} + \sum_i \frac{\partial(a_i(t,x)\cdot)}{\partial x_i} \tag{2.23}$$

exists and has the properties described earlier for fundamental solutions of parabolic equations with bounded coefficients (Theorem 2.2).

It follows easily from probabilistic considerations that

$$\int_{|x| < n} p_n(s,y;t,x) \, dx \le 1 \quad |y| < n \quad 0 \le s < t \le 1 \qquad (2.24)$$

and similarly

$$\int_{|y| < n} p_n(s,y;t,x) \, dy \le 1 \quad |x| < n \quad 0 \le s < t \le 1 \qquad (2.25)$$

Let $z \in \mathcal{R}^n$ be fixed and let ψ be a smooth nonnegative function with compact support such that z is in the interior of the support of ψ. Choose m so large that the support of ψ is contained in $|x| < m$. Define f_j $(j = m, m+1)$ as

$$f_j(s,y) = \int_{|x| < j} p_j(s,y;t,x)\psi(x) \, dx \qquad (2.26)$$

Then f_j satisfies the differential equation in Γ_n and for $s = t$ $f_j = \psi$ and $f_{m+1} \ge 0 = f_m$ on $|x| = m$. Hence by the maximum principle for partial differential equations [Ref. 33] $f_{m+1} \ge f_m$ in Γ_n. If we then replace ψ by a sequence of nonnegative functions which approximate the Dirac measure concentrated at z we obtain

$$p_{m+1} \ge p_m \quad \text{in} \quad \Gamma_m$$

since $|z| < m$ is arbitrary. We extend the definition of p_m to $\mathcal{R}^n$ by defining $p_m = 0$ if $|x|$ or $|y| > m$. Thus

$$0 \leq p_1 \leq p_2 \leq \cdots \leq p_m \leq p_{m+1} \leq \cdots$$

for $\forall x, y \in \mathcal{R}^n$, $t, s \in [0,1]$, $t > s$.

The sequence $\langle p_n \rangle$ has a finite limit a.e. (=p) but we must show that this limit is a solution of the parabolic equation, i.e.,

$$\mathcal{L}p = 0 \qquad\qquad (2.27)$$

We could in fact bound the sequence by the estimates obtained for parabolic equations with bounded coefficients since the constant α in Theorem 2.2 can be shown not to depend on the boundedness of b and c and the constant K can be simply related to the coefficients b and c.

Therefore for $t > s$ $x, y \in \mathcal{R}^n$

$$\lim_n p_n(s,y;t,x) = p(s,y;t,x) \quad \text{a.e.} \qquad\qquad (2.28)$$

is well defined (actually everywhere). It follows by the above that p is bounded and by Fatou's lemma

$$\int p(s,y;t,x) \, dx \leq 1 \qquad\qquad (2.29)$$

$$\int p(s,y;t,x) \, dy \leq 1 \qquad\qquad (2.30)$$

Consider a bounded domain $D \subset \mathcal{R}^n \otimes [0,1]$ and let $t > s$. Choose another bounded domain E such that $\bar{D} \subset E \subset \bar{E} \subset \mathcal{R}^n \otimes (s,1]$. Choose m so large that $\bar{E} \subset \Gamma_m$. By the Schauder-Barrar-Friedman interior estimates [Ref. 33]

$$\langle p_n \rangle \ , \ \left\langle \frac{\partial p_n}{\partial x_i} \right\rangle, \ \left\langle \frac{\partial^2 p_n}{\partial x_i \partial x_j} \right\rangle, \ \left\langle \frac{\partial p_n}{\partial t} \right\rangle$$

are uniformly bounded and equicontinuous in $\overline{D}$. Hence by the Ascoli theorem [Ref. 27] there exists a subsequence of $\langle p_n \rangle$ which together with its derivatives converges uniformly to $\overline{p}$ in $\overline{D}$. Therefore

$$\mathcal{L}\overline{p} \ = \ 0 \tag{2.31}$$

But

$$p_n \rightarrow p$$

Thus

$$\overline{p} \equiv p \quad \text{on} \quad \overline{D} \tag{2.32}$$

and since D was an arbitrary bounded domain

$$\mathcal{L}p \ = \ 0, \ x,y \in \mathcal{R}^n, \ s < t, \ s,t \in [0,1]$$

For completeness we should show that p is indeed a fundamental solution but this is straightforward and will therefore be omitted.

We should also verify that the assumptions on p to derive the forward Kolmogorov equation are valid. These assumptions are easily verified since

$$\sup_{0 \le t \le 1} E\left(x_t^2\right) < \infty$$

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and c is bounded.

The fact that p is a transition density, i.e.,

$$\int p \; = \; 1 \qquad\qquad (2.33)$$

as well as the uniqueness of the fundamental solution follow from our
proof for existence of the density. ∎

B. SOME REMARKS

Remark 1.

Theorem 2.1 is quite analogous to a result obtained by Eidelman
[Ref. 34]. For other results for linear parabolic equations with unbounded
coefficients the reader is referred to Krzyżański and Szybiak [Ref. 35],
S. Itô [Ref. 36], and Aronson and Besala [Ref. 37].

The method of proof given here seems to simplify somewhat the usual
construction of fundamental solutions by exploiting the probabilistic
interpretation of the parabolic equation.

Remark 2.

We have obtained the existence and uniqueness of a transition density
for a diffusion process which is the solution of a stochastic differential
equation and have shown that it is suitably differentiable when some of
the coefficients can be unbounded. Existence and uniqueness of the tran-
sition density is not true for arbitrary smooth but unbounded coefficients.

We construct an example from one-dimensional diffusion theory to show
that for some smooth but unbounded coefficients we will not obtain a
usual density function, i.e.,

$$\int p < 1 \qquad (2.34)$$

and the density function will not be unique.

Consider Brownian motion on the interval $[-\pi/2, \pi/2]$. It is intuitively clear (and not difficult to prove) that almost all sample paths of the Brownian motion will hit the boundaries $(x = -\pi/2, \pi/2)$. With suitable boundary conditions we can have a set of sample functions of positive probability absorbed at the boundaries.

Now we define a diffusion process on the extended real line by applying the smooth one-to-one transformation

$$y = \tan x \qquad (2.35)$$

to the Brownian motion on the interval $[-\pi/2, \pi/2]$. Therefore we have that a set of sample functions of positive probability of the new diffusion process is absorbed at the boundaries $(x = +\infty, -\infty)$. To clearly characterize this new diffusion process we compute its differential generator recalling that the differential generator for Brownian motion is

$$\frac{1}{2} \frac{\partial^2 f}{\partial x^2}$$

We obtain the differential generator for $y_t = \tan B_t$ by applying the chain rule for differentiation to

$$f(y) = f(\tan x) \qquad (2.36)$$

We obtain

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} \qquad (2.37)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial x}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial x^2} \qquad (2.38)$$

We use the following elementary results

$$\frac{d \tan x}{dx} = \sec^2 x$$

$$\frac{d \sec^2 x}{dx} = 2 \sec^2 x \tan x$$

$$\sec^2 x = \tan^2 x + 1$$

to obtain the differential generator

$$\frac{1}{2}\left(1 + y^2\right)^2 \frac{\partial^2 f}{\partial y^2} + y\left(1 + y^2\right)^2 \frac{\partial f}{\partial y} \qquad (2.39)$$

for the diffusion process on the real line. The diffusion process that we have constructed has the property that

$$\int dP < 1 \qquad (2.40)$$

where P is the transition measure for the process. Furthermore, by our construction it follows that the above differential generator (Eq. 2.39) does not correspond to a unique diffusion process.

Remark 3.

Random initial conditions for the stochastic differential equations cause no difficulty for determining a transition density if we assume that these random variables have a finite second moment (though finiteness a.s. suffices), that these random variables are independent of the Brownian motion, that the corresponding probability measures for these random variables have a density with respect to Lebesgue measure which is suitably differentiable, and that we change the various σ-fields and the probability measure P to include these random variables.

III. THE ABSOLUTE CONTINUITY OF MEASURES OF DIFFUSION PROCESSES

The main objective of this chapter is to obtain necessary and sufficent conditions for the absolute continuity of the measure of the solution of one stochastic differential equation with respect to the measure of the solution of another stochastic differential equation and to obtain an explicit expression for the density function in the space of continuous functions. The necessary and sufficient conditions derived here are analogous to the conditions obtained for Gaussian processes with independent increments [Ref. 12]. To obtain these conditions a characterization for the density function will be given which will indicate to some extent its structure.

The conditions for absolute continuity of the measures of solutions of stochastic differential equations have application in communication theory to the detection problem when the two hypotheses can be modeled by stochastic differential equations. The characterization of the density function should be useful for acquiring a better understanding of the nonlinear filtering problem and the associated stochastic optimal control problem.

The main result of this chapter is the following theorem.

Theorem 3.1. Let $\{x_t\}$ and $\{y_t\}$ satisfy

$$dx(t,\omega) = a(t,x(t,\omega)) \, dt + b(t,x(t,\omega)) \, dB(t,\omega) \qquad (3.1)$$

$$dy(t,\omega) = f(t,y(t,\omega)) \, dt + g(t,y(t,\omega)) \, dB(t,\omega) \qquad (3.2)$$

where $t \in [0,1]$, $x(0) = y(0)$ and the coefficients a, b, f and g

satisfy a global Lipschitz condition in the second variable and are continuous in the first variable and the diffusion matrices b and g have inverses for all values of their two variables. For

$$\mu_X \equiv \mu_Y \tag{3.3}$$

it is necessary and sufficient that

$$b^T b = g^T g \tag{3.4}$$

Corollary. For $\mu_Y \ll \mu_X$ the density function M_t can be written as a functional of only x_u, $0 \leq u \leq t$

$$M(x_u, \ 0 \leq u \leq t) = \exp\left[\int_0^t (f(s,x(s,\omega)) - a(s,x(s,\omega)))^T c^{-1}(s,x(s,\omega)) \ dx(s,\omega) \right.$$

$$- \int_0^t (f(s,x(s,\omega)) - a(s,x(s,\omega)))^T c^{-1}(s,x(s,\omega)) a(s,x(s,\omega)) \ ds$$

$$\left. - \frac{1}{2} \int_0^t (f(s,x(s,\omega)) - a(s,x(s,\omega)))^T c^{-1}(s,x(s,\omega))(f(s,x(s,\omega)) - a(s,x(s,\omega))) \ ds \right] \tag{3.5}$$

where

$$c = b^T b \tag{3.6}$$

We note that while this theorem is similar to the condition for Gaussian processes we have had to be more restrictive in our assumptions than in the Gaussian case since we have had to assume that the diffusion matrices have inverses.

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A. SOME SUFFICIENT CONDITIONS FOR ABSOLUTE CONTINUITY

To obtain necessary and sufficient conditions for the absolute conti-
nuity of measures of solutions of stochastic differential equations we
must first obtain an extension of the known sufficient conditions for
the absolute continuity of the measures. The extension is not difficult
using some techniques and results of Skorokhod [Ref. 13]. We first present
some of Skorokhod's results.

1. Skorokhod's Results

Lemma 3.1. If $b(t, x)$ and $g(t,x)$ are continuous in t and satisfy
a global Lipschitz condition in x, then the process $\left\{x_t^{(\alpha)}\right\}$ and
$\left\{y_t^{(\alpha)}\right\}$ defined for $\alpha = \{0, t_1, \ldots, t_\alpha = 1\}$ as

$$x^{(\alpha)}(t,\omega) = x^{(\alpha)}(t_k,\omega) + \int_{t_k}^{t} b\Big(t_k, x^{(\alpha)}(t_k,\omega)\Big)\, dB(s,\omega) \qquad (3.7)$$

$$y^{(\alpha)}(t,\omega) = y^{(\alpha)}(t_k,\omega) + \int_{t_k}^{t} g\Big(t_k, y^{(\alpha)}(t_k,\omega)\Big)\, dB(s,\omega) \qquad (3.8)$$

for $t \in [t_k, t_{k+1}]$ and

$$x^{(\alpha)}(0,\omega) = x(0,\omega)$$

$$y^{(\alpha)}(0,\omega) = y(0,\omega)$$

will for every $t \in [0,1] = T$ converge in probability to the solutions
of

$$x(t,\omega) = x(0,\omega) + \int_0^t b(s,x(s,\omega)) \, dB(s,\omega) \qquad (3.9)$$

$$y(t,\omega) = y(0,\omega) + \int_0^t g(s,y(s,\omega)) \, dB(s,\omega) \qquad (3.10)$$

as $\max_k(t_{k+1} - t_k) \to 0$ (i.e., the partitions become dense in the interval $[0,1]$).

Lemma 3.2. Let the finite-dimensional distributions of the processes $\zeta_n(t)$ and $\eta_n(t)$ converge weakly to the finite-dimensional distributions of the processes $\zeta(t)$ and $\eta(t)$, respectively, and let $P_1^{(n)}$, P_1, $P_2^{(n)}$, and P_2 be the measures in function space corresponding to the processes $\zeta_n(t)$, $\zeta(t)$, $\eta_n(t)$ and $\eta(t)$. Moreover, let the measure $P_1^{(n)}$ be absolutely continuous with respect to the measure $P_2^{(n)}$ for all n, let

$$\frac{dP_1^{(n)}}{dP_2^{(n)}} (x(t))$$

be the density of $P_1^{(n)}$ with respect to $P_2^{(n)}$, and let

$$\lim_{N \to \infty} \overline{\lim_{n \to \infty}} \, P\left\{ \left| \log \frac{dP_1^{(n)}}{dP_2^{(n)}} (x_n(t)) \right| > N \right\} = 0 \qquad (3.11)$$

Then the measure P_2 is absolutely continuous with respect to the measure P_1.

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<u>Lemma 3.3.</u> Let the processes $\zeta_n(t)$, $\zeta(t)$, $\eta_n(t)$, $\eta(t)$ satisfy the conditions of Lemma 3.2, and moreover, let $dP_2^{(n)}/dP_1^{(n)}$ exist. If $\zeta_n(t) \to \zeta(t)$, $\eta_n(t) \to \eta(t)$ and

$$\frac{dP_2^{(n)}}{dP_1^{(n)}} (\zeta_n(t)) \to \rho$$

in probability, then

$$\rho = \frac{dP_2}{dP_1} (\zeta(t)) \tag{3.12}$$

<u>Theorem 3.2.</u> Let $\zeta_1(t)$ and $\zeta_2(t)$ be Gaussian processes with independent increments such that

$$E\, e^{i(z,\zeta_j(t))} = \exp\left\{ - \frac{1}{2} (A_j(t)z,z) \right\} \tag{3.13}$$

In order that $\mu_{\zeta_2(t)}$ be absolutely continuous with respect to $\mu_{\zeta_1(t)}$ it is necessary and sufficient that $A_1(t) = A_2(t)$.

2. <u>New Result</u>

With these results of Skorokhod we are now in a position to extend the sufficient conditions for absolute continuity.

<u>Theorem 3.3.</u> If the processes $\{x_t\}$ and $\{y_t\}$ are defined as the solutions of the stochastic differential equations

continuous with respect to the measure $P_{(t,x)}^{(2)}$ corresponding to the process $\eta_{t,x}(\tau)$. We denote $x_t^{(\alpha)}$ as

$$x^{(\alpha)}(t,\omega) = x(t_k,\omega) + \zeta_{t,x}(t_k,\omega)^{(\alpha)} \qquad t \in [t_k, t_{k+1}] \qquad (3.19)$$

and by Lemma 3.1

$$x_t^{(\alpha)} \to x_t \quad \text{in probability} \quad t \in [0,1]$$

Since

$$\frac{dP^{(2)}}{dP^{(1)}} (\zeta(\tau)) \equiv 1 \qquad (3.20)$$

the conditions of Lemmas 3.2 and 3.3 are satisfied and therefore

$$\mu_Y \ll \mu_X$$

Similarly

$$\mu_X \ll \mu_Y \quad \blacksquare$$

B. CHARACTERIZATION OF THE DENSITY FUNCTION

1. Assumptions

Before considering necessary conditions for the absolute continuity of the probability measures we will make some assumptions as to the type of stochastic differential equations to be considered and assumptions on the coefficients in

$$dx(t,\omega) = b(t,x(t,\omega)) \, dB(t,\omega) \qquad (3.14)$$

$$dy(t,\omega) = g(t,y(t,\omega)) \, dB(t,\omega) \qquad (3.15)$$

where $b^T(t,x)b(t,x) = g^T(t,x)g(t,x)$ for all $t \in [0,1]$, $x \in \mathcal{R}^n$,
b and g satisfy a global Lipschitz condition in x and are
continuous in t and c^{-1} exists where $c = b^T b$, b and g are
$n \times n$ matrices and $\{x_t\}$, $\{y_t\}$ and $\{B_t\}$ are n-dimensional processes
then

$$\mu_X \equiv \mu_Y \qquad (3.16)$$

where μ_X and μ_Y correspond to the measures induced in function
space by $\{x_t\}$ and $\{y_t\}$ respectively.

Proof.

Define the homogeneous processes with independent increments ζ and
η:

$$\zeta_{t,x}(\tau) = \int_t^\tau b(t,x) \, dB(s,\omega) \qquad (3.17)$$

$$\eta_{t,x}(\tau) = \int_t^\tau g(t,x) \, dB(s,\omega) \qquad (3.18)$$

for $\tau \geq t$

it then follows from Skorokhod's theorem (Theorem 3.2) that $P^{(1)}_{(t,x)}$
corresponding to the process $\zeta_{t,x}(\tau)$ is for all t and x absolutely

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$$dx(t,\omega) = a(t,x(t,\omega)) \, dt + b(t,x(t,\omega)) \, dB(t,\omega) \qquad (3.1)$$

$$dy(t,\omega) = f(t,y(t,\omega)) \, dt + g(t,y(t,\omega)) \, dB(t,\omega) \qquad (3.2)$$

where

$$x(t,\omega) = (x_1(t,\omega),\ldots,x_n(t,\omega))^T \quad t \in [0,1]$$

$$y(t,\omega) = (y_1(t,\omega),\ldots,y_n(t,\omega))^T$$

$$B(t,\omega) = (B_1(t,\omega),\ldots,B_n(t,\omega))^T$$

$$a(t,x(t,\omega)) = \{a_i(t,x(t,\omega))\}$$

$$b(t,x(t,\omega)) = \{b_{ij}(t,x(t,\omega))\}$$

$$f(t,y(t,\omega)) = \{f_i(t,y(t,\omega))\}$$

$$g(t,y(t,\omega)) = \{g_{ij}(t,y(t,\omega))\}$$

Assumption. We will assume that the coefficients satisfy a global
   Lipschitz condition in the space variable and are continuous in the
   time variable and that the diffusion matrices b and g have
   inverses for $x \in \mathcal{R}^n$, $t \in T$. We will assume the interval of solution
   of these equations is $T = [0,1]$ and that $x(0) = y(0)$ although of
   course this last assumption on the initial conditions can be weakened
   to the case where these random variables have measures that are
   equivalent.

By the real valued function M_t on $C_n[0,t]$ we will mean the density function when $\mu_Y \ll \mu_X$ where μ_X and μ_Y correspond to the measures induced by the solutions of the respective stochastic differential equations above. By the Radon-Nikodym theorem we have for $\Lambda \in B(y_u, \; 0 \le u \le t)$.

$$\mu_Y(\Lambda) \; = \; \int_\Lambda M_t \; d\mu_X \qquad\qquad (3.21)$$

2. Uniform Integrability and Some Results From Martingale Theory

<u>Definition</u>. Let A be a subset of $L^1(\Omega, \mathfrak{F}, P)$. A is uniformly integrable if

$$\sup_{X \in A} \int_{\{|X| \ge n\}} |X(\omega)| P(d\omega) \to 0 \quad \text{as} \quad n \to \infty$$

The following results indicate the importance of uniform integrability.

<u>Theorem 3.4</u>. Let $\langle f_n \rangle$ be a sequence of integrable random variables that converge a.e. (or in probability) to a random variable f. Then f is integrable and the convergence of f_n to f takes place in the L^1 norm if and only if the f_n are uniformly integrable.

<u>Theorem 3.5 (Compactness Criterion of Dunford-Pettis)</u>. Let A be a subset of the space L^1. The following three properties are equivalent:

i. A is uniformly integrable.

ii. A is relatively compact in L^1 in the weak topology $\sigma\left(L^1, L^\infty\right)$.

iii. Every sequence of elements of A contains a subsequence that

converges in the sense of the topology $\sigma\left(L^1, L^\infty\right)$.

We state some well known results from martingale theory.

Proposition 3.1 (Jensen's Inequality). Let c be a convex mapping of $\mathcal{R}$ into $\mathcal{R}$ and let X be an integrable random variable such that the composition $c \circ X$ is integrable. The following inequality then holds

$$c \circ E[X|\mathcal{G}] \leq E[c \circ X|\mathcal{G}] \tag{3.22}$$

where $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$.

Theorem 3.6. Let $(X_t)_{t \in \mathcal{R}^+}$ be a right-continuous (or only separable) supermartingale where $\mathcal{R}^+$ is the positive half line

a. Suppose that

$$\sup_t E\left[X_t^-\right] < \infty$$

The random variables X_t then converge a.s. to an integrable random variable X_∞ as $t \to \infty$.

b. Suppose that the X_t are uniformly integrable. The above condition is then realized, the process $(X_t)_{t \in \mathcal{R}^+ \cup \{\infty\}}$ is a supermartingale, and the convergence takes place in the L^1 norm.

c. Suppose that the X_t are uniformly integrable and that the process (X_t) is a martingale. The process $(X_t)_{t \in \mathcal{R}^+ \cup \{\infty\}}$ is then a martingale.

The above results can be found in Meyer [Ref. 38]. The following

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results for necessary conditions were developed through discussions with S. Watanabe.

3. Functionals of Brownian Motion

Our first task is to give some general representation for the density function M_t. To accomplish this we will have to obtain some preliminary results on representations of martingales of Brownian motion.

<u>Definition.</u> The sub-σ-algebras $\mathcal{B}_t$ are defined as

$$\mathcal{B}_t = B(B_u, \ 0 \leq u \leq t)$$

<u>Theorem 3.7.</u> Any L_2 functional of n-dimensional Brownian motion can be represented by an infinite sum of stochastic integrals plus a constant.

<u>Corollary.</u> Every square integrable martingale of Brownian motion can be represented by an infinite sum of stochastic integrals plus a constant.

<u>Proof (Theorem).</u>

K. Itô [Ref. 39] proves the theorem for 1 dimensional Brownian motion, the n dimensional case follows from his results.

(Corollary). Recall $T = [0,1]$ for this chapter so let $t \in [0,1]$ and given

$$\sup_{t \, \in \, T} \ E\!\left(Y_t^2\right) < \infty$$

and $\{Y_t, \mathcal{B}_t\}$ is a martingale. By the martingale convergence theorem (Theorem 3.6) $\exists$ one and only one (up to equivalence) Y_1 such that for $t \in [0,1]$

$$Y_t = E[Y_1 | \boldsymbol{B}_t] \quad \text{a.s.} \tag{3.23}$$

By Itô's representation we can write

$$Y_1 = \sum_i \int_0^1 \Phi_i(s,\omega)\, dB_s + c \tag{3.24}$$

$$Y_t = E\left[\sum_i \int_0^1 \Phi_i(s,\omega)\, dB_s \,\Big|\, \boldsymbol{B}_t\right] + c$$

$$= \sum_i E\left[\int_0^t \Phi_i(s,\omega)\, dB_s \,\Big|\, \boldsymbol{B}_t\right] + \sum_i E\left[\int_0^1 \Phi_i(s,\omega)\, dB_s \,\Big|\, \boldsymbol{B}_t\right] + c \tag{3.25}$$

where the $\{\Phi_i(s,\omega)\}$ are n dimensional vectors which are $\boldsymbol{B}_s$ measurable for each s. (The integrand product above will mean throughout this chapter the usual inner product.) Since $\{Y_t, \boldsymbol{B}_t\}$ is a square integrable martingale the first sum is $\boldsymbol{B}_t$ measurable and the second sum is zero. Thus

$$Y_t = \sum_i \int_0^t \Phi_i(s,\omega)\, dB_s + c \quad \text{a.s.} \tag{3.26}$$

This was done for fixed t. Since the martingale is continuous we can apply the result to a countable dense T set and we therefore have for all t

$$Y_t = \sum_i \int_0^t \Phi_i(s,\omega)\, dB_s + c \quad \text{a.s.} \quad \blacksquare \tag{3.27}$$

In our attempt to characterize the density function M_t it is necessary

to relate the σ-algebras $B(x_u, 0 \le u \le t)$ and $B(y_u, 0 \le u \le t)$ to $B(B_u, 0 \le u \le t)$.

Lemma 3.4. For all t

$$B(y_u, 0 \le u \le t) = B(x_u, 0 \le u \le t) = B(B_u, 0 \le u \le t) \quad (3.28)$$

Proof.

We will prove only the last equality i.e.,

$$B(x_u, 0 \le u \le t) = B(B_u, 0 \le u \le t) \quad (3.29)$$

since the other proof follows similarly.

Since the coefficients of the stochastic differential equation are Lipschitz continuous we can apply the recursive formula (Picard iteration) that K. Itô uses to prove existence and uniqueness of the solution of stochastic differential equations. If we let $x^n(t,\omega)$ be the nth solution in the recursive procedure we clearly have

$$B\left(x_u^n, 0 \le u \le t\right) \subset B\left(B_u, 0 \le u \le t\right)$$

Since we are performing a countable operation we can pass to the limit and obtain

$$B(x_u, 0 \le u \le t) \subset B(B_u, 0 \le u \le t)$$

Conversely, since the diffusion matrix is nonsingular given any $x(t,\omega)$ we can determine $B(t,\omega)$ and therefore

$$B(x_u, \; 0 \leq u \leq t) \supset B(B_u, \; 0 \leq u \leq t) \qquad \blacksquare$$

We remark that this result is not always true as Itô and Nisio [Ref. 40] have indicated. They modify an example of Girsanov and show that

$$B(x_u, \; 0 \leq u \leq t) \not\subset B(B_u, \; 0 \leq u \leq t)$$

Conversely, trivial examples when the diffusion matrix is singular will show that the other σ-algebra inclusion is not always valid, i.e.,

$$B(x_u, \; 0 \leq u \leq t) \not\supset B(B_u, \; 0 \leq u \leq t)$$

We use the result of the preceding lemma (Lemma 3.4) to make the following assertions about the density function and subsequently to derive an expression for the density function.

<u>Lemma 3.5.</u> The density function M_t is a martingale of Brownian motion.

<u>Corollary.</u> The density function M_t is a continuous function of t.

<u>Proof (Lemma).</u>

Let $\Lambda \in B(y_u, \; 0 \leq u \leq t) = B(B_u, \; 0 \leq u \leq t)$ and let $\tau > t$

$$\int_{\Lambda} M_\tau \; d\mu_X = \int_{\Lambda} E[M_\tau | \; \mathcal{B}_t] \; d\mu_X = \int_{\Lambda} M_t \; d\mu_X \qquad (3.30)$$

Therefore $E[M_\tau | \mathcal{B}_t] = M_t$ a.s. μ_X and $\{M_t, \mathcal{B}_t\}$ is a martingale.

  <u>(Corollary).</u> If M_t is square integrable it follows by the representation for L^2 martingales of Brownian motion. Otherwise since

$0 \leq M_t < \infty$ a.s. μ_X we can approximate by L^2 martingales and obtain the result by the martingale convergence theorem (Theorem 3.6). ∎

4. Decomposition of Supermartingales

Doob in his development of martingale theory gave a unique decomposition of supermartingales for the discrete parameter case by a simple proof explicitly exhibiting the decomposition. The decomposition for continuous parameter right continuous supermartingales was finally completely solved a few years ago by P. A. Meyer. The problem is complicated a great deal by the continuous parameter and the decomposition is not valid for all continuous parameter supermartingales as it is for the discrete parameter case. Thus a number of definitions have to be given.

The concept of stopping time will play an important role in a number of the proofs given subsequently. A stopping time is defined as follows.

Definition. Let $(\Omega, \mathfrak{F})$ be a measure space and let $(\mathfrak{F}_t)_{t \in \mathfrak{R}^+}$ be an increasing family of sub-σ-fields of $\mathfrak{F}$. A positive random variable T defined on Ω is said to be a stopping time of the family $\{\mathfrak{F}_t\}$ if

$$\{\omega : T(\omega) \leq t\} \in \mathfrak{F}_t \quad \text{for every} \quad t \in \mathfrak{R}^+$$

A supermartingale is decomposed into the difference of a martingale and an increasing process. An important variety of increasing process is the following, since it will insure uniqueness of the decomposition.

Definition. An increasing process $\{A_t\}$ is said to be natural if for

every positive bounded, right-continuous martingale $\{Y_t\}$ we have

$$E\left[\int_0^t Y_s \, dA_s\right] = E\left[\int_0^t Y_{s-} \, dA_s\right] \qquad (3.31)$$

Remark.

If $\{A_t\}$ is continuous then it is natural.

Definition. Let $\{X_t\}$ be a right-continuous supermartingale relative
   to the family $\{\mathfrak{F}_t\}$ and let $\mathfrak{F}$ be the collection of all finite
   stopping times relative to this family (respectively, $\mathfrak{F}_a$ the
   collection of all stopping times bounded by a positive number a);
   $\{X_t\}$ is said to belong to the class (D) (respectively belong to
   the class (D) on the interval $[0,a]$) if the collection of random
   variables X_T $T \in \mathfrak{F}$ (respectively $T \in \mathfrak{F}_a$) is uniformly integrable.

   $\{X_t\}$ is said to belong to the class (DL), or locally to the
   class (D), if $\{X_t\}$ belongs to the class (D) on every interval
   $[0,a]$ $(0 \le a < \infty)$.

Definition. Let T be a stopping time relative to the family of σ-fields
   $(\mathfrak{F}_t)_{t\in\mathcal{R}^+}$. We denote by $\mathfrak{F}_T$ the collection of events $A \in \mathfrak{F}_\infty$ such
   that

$$A \cap \{T \le t\} \in \mathfrak{F}_t \quad \text{for every} \quad t \in \mathcal{R}^+$$

We will also have occasion in the subsequent work to use the optional
sampling theorem in the continuous parameter case.

Theorem 3.8. Suppose that for the supermartingale $\{X_t\}$ there exists an integrable random variable Y such that

$$X_t \geq E[Y|\mathfrak{F}_t] \quad \text{for each} \quad t \in \mathfrak{R}^+ \tag{3.32}$$

Let S and T be two stopping times such that $S \leq T$. The random variables X_S and X_T are then integrable, and we have the super-martingale inequality

$$X_S \geq E[X_T|\mathfrak{F}_S] \quad \text{a.s.} \tag{3.33}$$

The following theorem due to P. A. Meyer [Ref. 38] represents a complete solution to the decomposition of right-continuous, continuous parameter supermartingales. Its proof will not be included here, but it can be found in his book as can the preceding definitions.

Theorem 3.9. A right-continuous supermartingale $\{X_t\}$ has a decomposition

$$X_t = Y_t - A_t \tag{3.34}$$

where $\{Y_t\}$ denotes a right-continuous martingale and $\{A_t\}$ an increasing process if and only if $\{X_t\}$ belongs to the class (DL). There then exists a decomposition for which the process $\{A_t\}$ is natural, and this decomposition is unique.

5. An Expression for the Density Function

We have now the results necessary to characterize the density function M_t.

Theorem 3.10. If the density function, M_t, is strictly positive (a.s. μ_X) then there exists a function φ with $\varphi(s,\cdot)$ $\mathcal{B}_s$ measurable such that

$$M_t = \exp\left[\int_0^t \varphi(s,\omega)\,dB_s - \frac{1}{2}\int_0^t |\varphi(s,\omega)|^2\,ds\right] \qquad (3.35)$$

Proof.

By two previous lemmas we have that

i. M_t is a martingale of Brownian motion.

ii. M_t is a continuous function of t.

Recall that $t \in [0,1]$. Define stopping times T_n (with respect to $\mathcal{F}_t$) as

$$T_n = \inf\left\{t : M_t < \frac{1}{n} \text{ or } M_t > n\right\}$$

$$\qquad (3.36)$$

$$= 1 \text{ if the above set is empty.}$$

It is elementary to verify that $\langle T_n \rangle$ is a sequence of stopping times of the family $(\mathcal{B}_t)$. Clearly T_n is increasing with n and $T_n \uparrow 1$ a.s. μ_X.

We prove the latter. If $T_n < 1$ on $\Lambda : \mu_X(\Lambda) > 0$ then either $M_t > n$, $\forall n$ and $\int_\Lambda M_t\,d\mu_X = +\infty$ or $M_t < 1/n$, $\forall n$ and $M_t = 0$ on Λ both of which are contradictions. Note also that $\lim T_n = T$ is a stopping time since $\langle T_n \rangle$ is increasing. We truncate the density function M_t by use of the stopping times T_n

$$M_t^{(n)} = M_{t \wedge T_n} \qquad (3.37)$$

The pair $\left(M_t^{(n)}, \mathcal{B}_{T_n} \right)$ is a martingale by the optional sampling theorem. The function, $M_t^{(n)}$, is square integrable for each n. We now consider

$$X_t^{(n)} = \log M_t^{(n)} \qquad (3.38)$$

Since $M_t^{(n)}$ is bounded away from zero and bounded above for each n $E\left(|\log M_t^{(n)}| \right) < \infty$ so that $\log M_t^{(n)}$ is a supermartingale by Jensen's inequality. Since $X_t^{(n)}$ is bounded above and below it is easily verified that it is in class (D). Therefore we apply the Doob-Meyer decomposition and obtain

$$X_t^{(n)} = Y_t^{(n)} - A_t^{(n)} \qquad (3.39)$$

where $\left(Y_t^{(n)}, \mathcal{B}_{T_n} \right)$ is a martingale and $A_t^{(n)}$ is an increasing process. By Meyer's construction of the decomposition, the increasing process $A_t^{(n)}$ will be continuous if $X_t^{(n)}$ is continuous. Therefore (see remark on page 57) $A_t^{(n)}$ is natural and the decomposition is unique (Theorem 3.9). Since $A_t^{(n)}$ is continuous and $X_t^{(n)}$ is bounded it follows that $Y_t^{(n)}$ is bounded. Therefore the martingale $\left(Y_t^{(n)}, \mathcal{B}_{T_n} \right)$ is continuous and square integrable and can be represented by a sum of stochastic integrals (which, because of the continuity, can be summed to one stochastic integral) plus a constant term. Since $M_0 \equiv 1$ and $A_0 \equiv 0$ (by definition) it follows from Eq. (3.39) that Y_0 is zero and therefore the constant term is zero. Thus we can write $X_t^{(n)}$ in the form

$$X_t^{(n)} = \int_0^t \varphi^{(n)}(s,\omega)\, dB_s - \int_0^t dA_s^{(n)} \qquad (3.40)$$

Let $m > n$, then

$$M_t^{(n)} = M_t^{(m)} I_{\{t \le T_n\}} \qquad (3.41)$$

where $I_{\{t \le T_n\}}$ is the indicator function for the set $\{t \le T_n\}$.

By the uniqueness of the decomposition

$$A_t^{(n)} = A_t^{(m)} I_{\{t \le T_n\}} \qquad (3.42)$$

and thus

$$\varphi^{(n)}(s,\omega) = \varphi^{(m)}(s,\omega) I_{\{t \le T_n\}} \qquad (3.43)$$

By the assumptions on M_t and the definition of $\langle M_t^{(n)} \rangle$

$$M_t = \lim_{n \to \infty} M_t^{(n)} \quad \text{a.s.} \quad \mu_X \qquad (3.44)$$

thus we can define

$$A_t = \lim_n A_t^{(n)} \qquad\qquad \text{a.s.} \quad (3.45)$$

$$\varphi(t,\cdot) = \lim_n \varphi^{(n)}(t,\cdot) \qquad\qquad \text{a.s.} \quad (3.46)$$

$$M_t = \exp\left[\int_0^t \varphi(s,\omega)\, dB_s - \int_0^t dA_s\right] \qquad (3.47)$$

Let $dz(t,\omega) = \varphi(t,\omega)\, dB_t - dA_t$ and apply the stochastic differential rule (Theorem 1.2) of K. Itô to

$$\exp [z_t]$$

The term dA_t causes no difficulty because A_t is of bounded variation. Thus

$$M_t = 1 + \int_0^t M_s \varphi(s,\omega) \, dB_s + \frac{1}{2} \int_0^t M_s |\varphi(s,\omega)|^2 \, ds - \int_0^t M_s \, dA_s \quad (3.48)$$

By definition of the density function

$$E[M_t - 1] = 0 \qquad (3.49)$$

and $\{M_t - 1, \mathscr{F}_t\}$ is a martingale with

$$E \left[\int_0^t M_s \varphi(s,\omega) \, dB_s + \frac{1}{2} \int_0^t M_s |\varphi(s,\omega)|^2 \, ds - \int_0^t dA_s \right] = 0 \qquad (3.50)$$

Since the terms inside the expectation are a martingale of Brownian motion this implies

$$dA_s = \frac{1}{2} |\varphi(s,\omega)|^2 \, ds \qquad (3.51)$$

$$M_t = \exp \left[\int_0^t \varphi(s,\omega) \, dB_s - \frac{1}{2} \int_0^t |\varphi(s,\omega)|^2 \, ds \right] \blacksquare \qquad (3.52)$$

C. NECESSARY AND SUFFICIENT CONDITIONS FOR ABSOLUTE CONTINUITY

We now have established the preliminaries necessary to obtain the main result of this chapter, that is, necessary and sufficient conditions for the absolute continuity of measures which are generated by solutions

of stochastic differential equations.

<u>Theorem 3.1.</u> Let $\{x_t\}$ and $\{y_t\}$ satisfy

$$dx(t,\omega) \; = \; a(t,x(t,\omega)) \; dt \; + \; b(t,x(t,\omega)) \; dB(t,\omega) \tag{3.1}$$

$$dy(t,\omega) \; = \; f(t,y(t,\omega)) \; dt \; + \; g(t,y(t,\omega)) \; dB(t,\omega) \tag{3.2}$$

where $t \in [0,1]$, $x(0) = y(0)$ and the coefficients a, b, f, g
satisfy a global Lipschitz condition in the second variable and are
continuous in the first variable and the diffusion matrices b and
g have inverses for all values of their two variables. For $\mu_X \equiv \mu_Y$
it is necessary and sufficient that

$$b^T b \; = \; g^T g \tag{3.4}$$

<u>Corollary.</u> For $\mu_Y \ll \mu_X$ the density function M_t can be written as
a functional of only x_u, $0 \le u \le t$

$$
\begin{aligned}
M(x_u, \; 0 \le u \le t) \; = \; \exp\Bigg[&\int_0^t (f(s,x(s,\omega)) - a(s,x(s,\omega)))^T c^{-1}(s,x(s,\omega)) \; dx(s,\omega) \\
&- \int_0^t (f(s,x(s,\omega)) - a(s,x(s,\omega)))^T c^{-1}(s,x(s,\omega)) a(s,x(s,\omega)) \; ds \\
&- \frac{1}{2} \int_0^t (f(s,x(s,\omega)) - a(s,x(s,\omega)))^T c^{-1}(s,x(s,\omega))(f(s,x(s,\omega)) - a(s,x(s,\omega))) \; ds \Bigg]
\end{aligned}
\tag{3.5}
$$

where

$$c = b^T b \tag{3.6}$$

Proof of Theorem (Necessity).

By the preceding theorem we know that the density function M_t is of the following form

$$M_t = \exp\left[\int_0^t \varphi(s,\omega)\, dB_s - \frac{1}{2}\int_0^t |\varphi(s,\omega)|^2\, ds\right] \tag{3.52}$$

We will now characterize the vector function φ by applying the stochastic differential rule. For any $h \in C^2$, $h \in L^2(dt \times dP)$, $h : \mathcal{R}^n \to \mathcal{R}$

$$\lim_{t \downarrow 0} \frac{1}{t} E\left[\exp\left[\int_0^t \varphi_s\, dB_s - \frac{1}{2}\int_0^t |\varphi_s|^2\, ds\right](h(x_t) - h(x_0))\right] = \lim_{t \downarrow 0} \frac{1}{t} E\left[h(y_t) - h(y_0)\right]$$

$$= \frac{g^T g}{2} h'' + f h' \tag{3.53}$$

$$\lim_{t \downarrow 0} \frac{1}{t} E\left[M_t(h(x_t) - h(x_0))\right] \tag{3.54}$$

$$= \lim_{t \downarrow 0} \frac{1}{t} E\left[(M_t - 1)(h(x_t) - h(x_0)) + (h(x_t) - h(x_0))\right] \tag{3.55}$$

$$\lim_{t \downarrow 0} \frac{1}{t} E\left[(h(x_t) - h(x_0))\right] = \frac{b^T b}{2} h'' + a h' \tag{3.56}$$

where

$$b^T b h'' = \sum_{i,j} \frac{\partial^2 h}{\partial x_i \partial x_j} \sum_k b_{ik} b_{kj}$$

We recall (somewhat heuristically here) the behavior of products of ordinary and stochastic integrals as $t \downarrow 0$

$$\int_0^t \alpha_s \, ds \int_0^t \beta_\tau \, dB_\tau \sim o(t)$$

$$\int_0^t \alpha_s \, ds \int_0^t \beta_\tau \, dB_\tau \sim o(t)$$

$$\int_0^t \alpha_s \, dB_s \int_0^t \beta_\tau \, dB_\tau \sim \int_0^t \alpha_s \beta_s \, ds \sim \mathcal{O}(t)$$

The above results are proved by K. Itô in his derivation of the stochastic differential rule. Recall also

$$M_t - 1 = \int_0^t M_s \varphi_s \, dB_s \qquad (3.57)$$

and

$$h(x_t) - h(x_0) = \int_0^t h_s' a_s \, ds + \frac{1}{2} \int_0^t h_s'' b_s^T b_s \, ds + \int_0^t h_s' b_s \, dB_s \qquad (3.58)$$

$$\lim_{t \downarrow 0} \frac{1}{t} E\left[\int M_s \varphi_s \, dB_s \left(\int h_s' a_s \, ds + \frac{1}{2} \int h_s'' b_s^T b_s \, ds + \int h_s' b_s \, dB_s \right) \right] = b\varphi h' \qquad (3.59)$$

Therefore

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$$\lim_{t \downarrow 0} \frac{1}{t} E\left[M_t(h(x_t) - h(x_0))\right]$$

$$= \frac{b^T b}{2} h'' + ah' + b\varphi h' \qquad (3.60)$$

$$= \frac{g^T g}{2} h'' + fh' \qquad (3.53)$$

The last equality is from our initial calculation (Eq. 3.53). Since this was done for arbitrary $h \in L^2(dt \times dP)$, $h \in C^2$ we have

$$b^T b = g^T g$$

$$f = a + b\varphi$$

Our hypothesis that $\mu_X \ll \mu_Y$ insures that $0 < M_t < \infty$ a.s. μ_X.

Proof (Sufficiency).

The proof is elementary so we will only sketch it. First define a correspondence between measures and solutions of stochastic differential equations (where we have suppressed the arguments of the coefficients)

$$\mu_1 \sim b\, dB_t$$

$$\mu_2 \sim a\, dt + b\, dB_t$$

$$\mu_3 \sim g\, dB_t$$

$$\mu_4 \sim f\, dt + g\, dB_t$$

From the usual results for absolute continuity (Theorem 1.3) we have

$$\mu_1 \equiv \mu_2$$

$$\mu_3 \equiv \mu_4$$

and from our extensions we have

$$\mu_1 \equiv \mu_3$$

It easily follows from the transitivity of the absolute continuity relation that

$$\mu_2 \equiv \mu_4 \qquad \blacksquare$$

Proof of Corollary.

We note from our proof of necessity for the theorem that

$$f = a + b\varphi \tag{3.61}$$

so

$$\varphi = b^{-1}(f - a) \tag{3.62}$$

and

$$dB_t = b^{-1}(dx_t - a\,dt) \tag{3.63}$$

Upon substituting in the expression for the density the above quantities

for φ and dB_t we obtain the result. ∎

IV. THE CONDITIONAL PROBABILITY DENSITY

A. INTRODUCTION AND MAIN RESULT

Consider a vector Markov process $\{x_t\}$ satisfying

$$dx(t,\omega) = a(t,x(t,\omega)) \, dt + b(t,x(t,\omega)) \, dB(t,\omega) \qquad (4.1)$$

whose states x_t cannot be observed directly but only through the noisy observations

$$dy(t,\omega) = g(t,x(t,\omega),y(t,\omega)) \, dt + h(t,y(t,\omega)) \, d\tilde{B}(t,\omega) \qquad (4.2)$$

where $t \in [s,1]$ $x(s) = \alpha$, $y(s) = 0$

$$x(t,\omega) = (x_1(t,\omega), x_2(t,\omega), \ldots, x_n(t,\omega))^T$$

$$y(t,\omega) = (y_1(t,\omega), y_2(t,\omega), \ldots, y_m(t,\omega))^T$$

a is a vector in $\mathcal{R}^n$, b is an $n \times n$ matrix, g is a vector in $\mathcal{R}^m$ and h is an $m \times m$ matrix and $\{B_t\}$ and $\{\tilde{B}_t\}$ are independent Brownian motions in $\mathcal{R}^n$ and $\mathcal{R}^m$ respectively.

Many control and communication problems associated with these stochastic differential equations require knowledge of the conditional probability density $p(x,t|\alpha,s,y_u, \ s \leq u \leq t)$ which is the probability density that $x_t = x$ given the observations $B(y_u, \ s \leq u \leq t)$ and that $x(s) = \alpha$.

In this chapter we shall show that such a conditional probability density function exists and shall give a formula for it (Eq. 4.15). This formula is difficult to evaluate as it stands and therefore we shall obtain

from it a stochastic differential equation that will describe the evolution in time of the conditional probability density. This equation is a non-linear equation and except in certain very special cases [Ref. 41] no explicit solutions are known.

The main result of this chapter is the following theorem.

<u>Theorem 4.1.</u> Let $\{x_t\}$ and $\{y_t\}$ satisfy

$$dx(t,\omega) = a(t,x(t,\omega)) \, dt + b(t,x(t,\omega)) \, dB(t,\omega) \qquad (4.1)$$

$$dy(t,\omega) = g(t,x(t,\omega),y(t,\omega)) \, dt + h(t,y(t,\omega)) \, d\tilde{B}(t,\omega) \qquad (4.2)$$

where $x_s = \alpha$, $y_s = 0$, $t \in [s,1]$, $\{B_t\}$ and $\{\tilde{B}_t\}$ are independent Brownian motions in $\mathfrak{R}^n$ and $\mathfrak{R}^m$ respectively and

i) The diffusion matrix $b(t,x)$ is Hölder continuous in t, globally Lipschitz continuous in x and globally bounded. Moreover, the symmetric matrix c $\left(c = b^T b\right)$ is strictly positive definite. The terms

$$\frac{\partial c_{ij}(t,x)}{\partial x_i} \, , \, \frac{\partial^2 c_{ij}(t,x)}{\partial x_i \partial x_j} \qquad i,j = 1,2,\dots,n$$

are globally Lipschitz continuous in x, continuous in t, and globally bounded.

ii) The transfer (drift) vector $a(t,x)$ is continuous in t and globally Lipschitz continuous in x. The terms

$$\frac{\partial a_i(t,x)}{\partial x_i} \qquad i = 1,2,\ldots,n$$

are globally Lipschitz continuous in x and continuous in t.

iii) The transfer (drift) vector $g(t,x,y)$ and the diffusion matrix $h(t,y)$ satisfy a global Lipschitz condition in x and y and are continuous in t. Moreover, the symmetric matrix f $\left(f = h^T h\right)$ is strictly positive definite.

Then the conditional probability density $p(x,t|\alpha,s,y_u, \ s \leq u \leq t)$ exists and satisfies the following stochastic differential equation

$$dp_t = \left(\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 (c_{ij}(t,x)p_t)}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial (a_i(t,x)p_t)}{\partial x_i} \right) dt$$

$$+ (g_t - \hat{g}_t)^T f_t^{-1}(dy_t - \hat{g}_t \ dt) \qquad (4.3)$$

where

$$p_t = p(x,t|\alpha,s,y_u, \ s \leq u \leq t) \qquad (4.4)$$

$$g_t = g(t,x,y_t) \qquad (4.5)$$

$$f_t = f(t,y_t) \qquad (4.6)$$

$$\hat{g}_t = \frac{\int \psi_t g(t,x_t,y_t) \ d\mu_X}{\int \psi_t \ d\mu_X} \qquad (4.7)$$

$$\psi_t = \psi_t(x_u, y_u, \ s \le u \le t) = \exp\left[\int_s^t g^T(u, x_u, y_u) f^{-1}(u, y_u) \ dy_u\right.$$

$$\left. - \frac{1}{2} \int_s^t g^T(u, x_u, y_u) f^{-1}(u, y_u) g(u, x_u, y_u) \ du\right] \tag{4.8}$$

Stratonovich [Ref. 2] was apparently the first to consider the non-linear filtering problem. His equation for the evolution of the conditional probability differs from our result because his stochastic integrals are not interpreted in the K. Itô sense [cf. Ref. 9]. Subsequent to Stratonovich, Kushner [Refs. 4,5,6], Kashyap [Ref. 3], Bucy [Ref. 7], and Mortensen [Ref. 8] have also discussed this problem. We rigorously derive the stochastic equation for the conditional probability under weaker assumptions than has been done [Refs. 6,8]. The style of proof that we shall give was first used by Mortensen but our results are extensions of Mortensen's work by allowing coefficients of the stochastic differential equations to be unbounded and assuming a more general form for the stochastic differential equations. Recently Shiryaev [Ref. 42] has sketched a proof of the equation for the evolution of the conditional probability density for a more general problem than we have considered but he did not indicate the assumptions that he made so we cannot compare the results. We also derive an equation for the conditional moments.

B. PROOF OF THE MAIN RESULT

Since the proof is long and quite detailed we shall first outline the major steps in it. We first prove the existence of the conditional probability measure and give an expression for it (Lemma 4.1). Recalling our result from Chapter 2 that the transition density exists (Lemma 2.3)

we prove that the conditional probability density exists and obtain an expression for it which involves the transition density (Lemma 4.2). From this expression for the conditional probability density we shall obtain a stochastic equation for the evolution of this conditional probability density. To do this we shall need a Fubini-type result for a stochastic integral and an ordinary integral (Lemma 4.3). Having this we then establish a stochastic equation for the unnormalized conditional probability density (Lemma 4.4). Finally we combine this result with the differential for the normalization constant to obtain the main result.

We shall need to define various quantities and symbols. Our fundamental σ-algebra in this chapter is the augmented σ-algebra on the space of continuous functions that take values in $\mathcal{R}^{n+m}$. We denote it by

$$\mathfrak{F}_t = B(B_u, \tilde{B}_u, \ s \leq u \leq t) \overset{\bullet}{=} B(B_u, \ s \leq u \leq t) \oplus B(\tilde{B}_u, \ s \leq u \leq t) \tag{4.9}$$

where $\overset{\bullet}{=}$ means equality up to sets of measure zero. Our fundamental measure P then will be a measure on this space of continuous functions $C_{n+m}[s,t]$. We define the following measures. Let μ_X be the measure on the space $C_n[s,1]$ induced by the solution of

$$dx_t = a(t,x_t)\,dt + b(t,x_t)\,dB_t \tag{4.1}$$

Let ρ_X be the measure induced on the space $C_n[s,1]$ by the solution of

$$dx_t = b(t,x_t)\,dB_t \tag{4.10}$$

Let ρ_Y be the measure on the space $C_m[s,1]$ induced by the solution of

$$dy(t,\omega) = h(t,y(t,\omega)) \, d\tilde{B}(t,\omega) \qquad (4.11)$$

Let $\mu_{X \times Y}$ be the measure on the space $C_{n+m}[s,1]$ induced by the solution of

$$dx_t = a(t,x_t) \, dt + b(t,x_t) \, dB_t \qquad (4.1)$$

$$dy_t = g(t,x_t,y_t) \, dt + h(t,y_t) \, d\tilde{B}_t \qquad (4.2)$$

We also define the real-valued functions φ_t and ψ_t using Girsanov's theorem (Theorem 1.3)

$$\varphi_t(x_u, \ s \leq u \leq t) = \exp\left[\int_s^t a^T(u,x_u)c^{-1}(u,x_u) \, dx_u\right.$$

$$\left. - \frac{1}{2} \int_s^t a^T(u,x_u)c^{-1}(u,x_u)a(u,x_u) \, du\right] \qquad (4.12)$$

$$\psi_t(x_u,y_u, \ s \leq u \leq t) = \exp\left[\int_s^t g^T(u,x_u,y_u)f^{-1}(u,y_u) \, dy_u\right.$$

$$\left. - \frac{1}{2}\int_s^t g^T(u,x_u,y_u)f^{-1}(u,y_u)g(u,x_u,y_u) \, du\right] \qquad (4.8)$$

where

$$\varphi_1 = \frac{d\mu_X}{d\rho_X}$$

and

$$\varphi_1\psi_1 = \frac{d\mu_{XxY}}{d(\rho_X \times \rho_Y)}$$

By the martingale property of the density function (Lemma 3.5) we can write for $\Lambda \in B(x_u, s \leq u \leq t)$

$$\mu_X(\Lambda) = E_{\mu_X}[I_\Lambda] = E_{\rho_X}[I_\Lambda\varphi_1]$$

$$= E_{\rho_X}[I_\Lambda E_{\rho_X}[\varphi_1|B(x_u, s \leq u \leq t)]]$$

$$= E_{\rho_X}[I_\Lambda\varphi_t] = \int_\Lambda \varphi_t\, d\rho_X \qquad (4.13)$$

and for $\Gamma \in B(x_u, y_u, s \leq u \leq t)$

$$\mu_{XxY}(\Gamma) = E_{\mu_{XxY}}[I_\Gamma] = E[I_\Gamma\varphi_1\psi_1]$$

$$= E[I_\Gamma E[\varphi_1\psi_1|B(x_u, y_u, s \leq u \leq t)]]$$

$$= E[I_\Gamma\varphi_t\psi_t] = \int_\Gamma \varphi_t\psi_t\, d(\rho_X \times \rho_Y) \qquad (4.14)$$

For notational simplicity we have not explicitly indicated the initial conditions $x(s)$ and $y(s)$. Our basic probability space Ω will be

induced from the independent Brownian motions $\{B_t\}$ and $\{\tilde{B}_t\}$ and be on $C_{n+m}[s,1]$. Since all of our random functions depend only on $B_t, \tilde{B}_t, x_t, y_t (t \in [s,1])$ it will suffice for our fundamental measure P to consider $\rho_X \times \rho_Y$.

1. An Expression for the Conditional Probability Measure

We first discuss the conditional probability measure in function space. The existence of this conditional probability measure follows from the usual arguments of conditional expectation, i.e., via the Radon-Nikodym theorem. We now derive an expression for the conditional probability.

Lemma 4.1. The conditional probability measure in function space for all $t \in T = [s,1]$ is given by

$$P(\Lambda, t | \alpha, s, y_u, \ s \le u \le t) = \frac{E_{\rho_X}[I_\Lambda \varphi_t \psi_t]}{E_{\rho_X}[\varphi_t \psi_t]} \qquad (4.15)$$

where $\Lambda \in B(x_t)$.

Proof.

Fix $t \in [s,1]$. Let ϕ_t and Ω_t be the empty set and the whole space in $C_n[s,1]$. Define the σ-algebra $\mathcal{G}_t$ on $C_{n+m}[s,1]$ as

$$\mathcal{G}_t = \{\phi_t, \Omega_t\} \oplus B(y_u, \ s \le u \le t) \qquad (4.16)$$

Let

$$\Pi_Y \mathcal{G}_t = B(y_u, \ s \le u \le t) \qquad (4.17)$$

which is a σ-algebra on $C_m[s,1]$. We have for $\Gamma \in \mathcal{G}_t$

$$P_\Gamma(\Lambda,t|\alpha,s) = \int_\Gamma I_\Lambda(x_t) \, d\mu_{XxY} = \int_\Gamma P(\Lambda,t|\alpha,s,\mathcal{G}_t) \, d\mu_{XxY}$$

$$= \int_\Gamma I_\Lambda(x_t)\varphi_t\psi_t \, d(\rho_X \times \rho_Y) = \int_\Gamma P(\Lambda,t|\alpha,s,\mathcal{G}_t)\varphi_t\psi_t \, d(\rho_X \times \rho_Y)$$

$$(4.18)$$

We have

$$E[P(\Lambda,t|\alpha,s,\mathcal{G}_t)\varphi_t\psi_t|\mathcal{G}_t] = E[I_\Lambda\varphi_t\psi_t|\mathcal{G}_t] \quad \text{a.s.} \quad (4.19)$$

$$P(\Lambda,t|\alpha,s,\mathcal{G}_t)E[\varphi_t\psi_t|\mathcal{G}_t] = E[I_\Lambda\varphi_t\psi_t|\mathcal{G}_t] \quad \text{a.s.} \quad (4.20)$$

The last equation follows since $P(\Lambda,t|\alpha,s,\mathcal{G}_t)$ is a $\mathcal{G}_t$ measurable function. We now consider the product measure $\rho_X \times \rho_Y$ and take iterated integrals using Tonelli's theorem. Since both sides of Eq. (4.20) are zero for the integral on $\emptyset_t$, we have only the integral on Ω_t and for the R.H.S. of Eq. (4.20) it is

$$\int E_{\rho_Y}[I_\Lambda\varphi_t\psi_t|\Pi_Y\mathcal{G}_t] \, d\rho_X = E[I_\Lambda\varphi_t\psi_t|\mathcal{G}_t = \Omega_t \oplus \Pi_Y\mathcal{G}_t] \quad (4.21)$$

We consider the variable of integration on ρ_X as x_u, $s \leq u \leq t$. So for a fixed value of this variable the product $I_\Lambda\varphi_t$ is a constant and we have

$$E_{\rho_Y}[I_\Lambda\varphi_t\psi_t|\Pi_Y\mathcal{G}_t] = I_\Lambda\varphi_t E_{\rho_Y}[\psi_t|\Pi_Y\mathcal{G}_t] \quad \text{a.s.}$$

$$= I_\Lambda\varphi_t\psi_t \quad \text{a.s.} \quad (4.22)$$

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The last equality follows since ψ_t is a density function and there-
fore measurable with respect to $\Pi_Y \mathcal{S}_t$. Since φ_t and ψ_t are continuous
in $\{x_u, \ s \le u \le t\}$ and I_Λ can be approximated in measure by continuous
functions we need only consider polynomials with rational coefficients
on $[s,t]$. So we only have a countable number of values for the process
$\{x_t\}$ and since the uniform integrability condition can be easily verified
for these functions we therefore have

$$\int E_{\rho_Y}[I_\Lambda \varphi_t \psi_t | \Pi_Y \mathcal{S}_t] \ d\rho_X \ = \ \int I_\Lambda \varphi_t E_{\rho_Y}[\psi_t | \Pi_Y \mathcal{S}_t] \ d\rho_X \ = \ \int I_\Lambda \varphi_t \psi_t \ d\rho_X$$

$$(4.23)$$

Therefore

$$P(\Lambda, t | \alpha, s, \Pi_Y \mathcal{S}_t) \ = \ \frac{E_{\rho_X}[I_\Lambda \varphi_t \psi_t]}{E_{\rho_X}[\varphi_t \psi_t]} \quad \text{a.s.} \qquad (4.24)$$

This was done for fixed t. Since the product $\varphi_t \psi_t$ can be represented
by a stochastic integral (cf. Eq. (1.22)) it is continuous in t and
of course the sample paths are continuous. Using the above result on
a dense T set we can take a limit and easily verify uniform integra-
bility.

Therefore we can obtain the result for all $t \in T$

$$P(\Lambda, t | \alpha, s, y_u, \ s \le u \le t) \ = \ \frac{E_{\rho_X}[I_\Lambda \varphi_t \psi_t]}{E_{\rho_X}[\varphi_t \psi_t]} \quad \text{a.s.} \qquad (4.25)$$

2. An Expression for the Conditional Probability Density

In Chapter 2 we proved that the process $\{x_t\}$ has a transition density with respect to Lebesgue measure and that it could be represented as the fundamental solution of Kolmogorov's equations. To prove that the conditional density exists we necessarily use the existence of the transition density for the process $\{x_t\}$. The following representation for the conditional probability density has been obtained by others. Bucy [Ref. 7] obtained it without including a proof (assuming that the transition density existed) and Mortensen [Ref. 8] obtained it under more restrictive assumptions.

Lemma 4.2. Let p_X be the transition density corresponding to the process $\{x_t\}$. The conditional probability density with respect to Lebesgue measure exists and is given by the following equation

$$p(x,t|\alpha,s,y_u, \; s \leq u \leq t) = \frac{E_{\mu_X}[\psi_t|x_t = x]p_X(s,\alpha;t,x)}{E_{\mu_X}[\psi_t]} \qquad (4.26)$$

where $t \in (s,1]$.

Proof.

Fix $t \in (s,1]$. Let $\Lambda \in B(x_t)$.

$$E_{\rho_X}[I_\Lambda \varphi_t \psi_t] = E_{\mu_X}[I_\Lambda \psi_t] = E_{\mu_X}[E_{\mu_X}[I_\Lambda(x_t)\psi_t|x_t]]$$

$$= E_{\mu_X}[I_\Lambda(x_t)E_{\mu_X}[\psi_t|x_t]]$$

$$= \int I_\Lambda(x_t)E_{\mu_X}[\psi_t|x_t] \, d\mu_X \qquad (4.27)$$

The last integral reduces to an integral over $\mathcal{R}^n$ since the integrand depends only on x_t. So we have the measure $P_X(s,\alpha;t,dx)$ induced from $\dot{d}\mu_X$.

We therefore have

$$E_{\mu_X}[I_\Lambda \psi_t] = \int_{\mathcal{R}^n} I_\Lambda(x) E_{\mu_X}[\psi_t | x_t = x] P_X(s,\alpha;t,dx) \qquad (4.28)$$

Since P_X is absolutely continuous with respect to Lebesgue measure we have

$$E_{\mu_X}[I_\Lambda \psi_t] = \int_{\mathcal{R}^n} I_\Lambda(x) E_{\mu_X}[\psi_t | x_t = x] p_X(s,\alpha;t,x)\, dx \qquad (4.29)$$

Since this was done for arbitrary $\Lambda \in B(x_t)$ by the Radon-Nikodym theorem we have

$$p_X(x,t|\alpha,s,y_u, \; s \le u \le t) = \frac{E_{\mu_X}[\psi_t | x_t = x] p_X(s,\alpha;t,x)}{E_{\mu_X}[\psi_t]} \quad \text{a.e.} \quad dx \qquad (4.30)$$

The above result was obtained for fixed t. It follows easily that the result is true for a countable dense T set, say S. For arbitrary $t \in T \; \exists$ sequence $\langle t_n \rangle$ with $t_n \to t$ and $t_n \in S$. Let $\Lambda_a = \{x_t < a\}$. For sets of this form, Λ_a, using the continuity of the sample paths $\{x_t\}$ and the uniform integrability of the sequence $\langle p_n \rangle$ (corresponding to t_n) we can establish the result. For an arbitrary set $\Lambda \in B(x_t)$ we can approximate by sets of the form Λ_a. Therefore we have the result for all $t \in (s,1]$. ∎

3. A Stochastic Equation for the Conditional Probability Density

We shall now proceed to derive a stochastic equation for the evolution in time of the conditional probability density. To obtain this equation we first apply the stochastic differential rule to ψ_t in the function $E_{\mu_X}[\psi_t | x_t = x]$ and obtain a Fubini-type result for an ordinary and a stochastic integral.

Lemma 4.3. For the function

$$E_{\mu_X}[\psi_t | x_t = x] = E_{\mu_X}\left[1 + \int_s^t \psi_u g_u^T f_u^{-1} \, dy_u \Big| x_t = x\right] \qquad (4.31)$$

in the conditional probability density, the following result is valid a.s.

$$E_{\mu_X}\left[\int_s^t \psi_u g_u^T f_u^{-1} \, dy_u \Big| x_t = x\right] = \int_s^t E_{\mu_X}[\psi_u g_u^T f_u^{-1} | x_t = x] \, dy_u \qquad (4.32)$$

where g_u and f_u^{-1} correspond to $g(u,x(u,\omega),y(u,\omega))$ and $f^{-1}(u,y(u,\omega))$ respectively.

Proof.

Let

$$\Omega_M = \{\omega : \sup_{s \le t \le 1} |x_t| < M, \quad \sup_{s \le t \le 1} |y_t| < M\} \qquad (4.33)$$

Since $\{x_t\}$ and $\{y_t\}$ have continuous sample paths

$$P\left(\bigcup_{M} \Omega_M\right) = 1 = \lim_{M \to \infty} P(\Omega_M) \qquad (4.34)$$

The last equality follows since $\Omega_{M+1} \supset \Omega_M$. Let $\Lambda \in B(x_t)$. By the boundedness of the integrand on Ω_M we can define the stochastic integral as a limit of step function approximations and the interchange of order of integrations is clearly valid for step functions since we are given $B(y_u, \; s \leq u \leq t)$. Therefore we have

$$\int_{\Lambda \cap \Omega_M} \int_s^t \psi_u g_u^T f_u^{-1} \, dy_u \, d\mu_X = \int_s^t \int_{\Lambda \cap \Omega_M} \psi_u g_u^T f_u^{-1} \, d\mu_X \, dy_u \quad \text{a.s.} \qquad (4.35)$$

Recalling that the stochastic integral is $\psi_t - 1$ we can take the following limit and have the desired equality

$$\lim_{M \to \infty} \int_{\Lambda \cap \Omega_M} \int_s^t \psi_u g_u^T f_u^{-1} \, dy_u \, d\mu_X = \int_\Lambda \int_s^t \psi_u g_u^T f_u^{-1} \, dy_u \, d\mu_X \quad \text{a.s.} \quad \rho_Y \qquad (4.36)$$

We therefore have

$$\lim_{M \to \infty} \int_{\Lambda \cap \Omega_M} \int_s^t \psi_u g_u^T f_u^{-1} \, dy_u \, d\mu_X = \lim_{M \to \infty} \int_s^t \int_{\Lambda \cap \Omega_M} \psi_u g_u^T f_u^{-1} \, d\mu_X \, dy_u \quad \text{a.s.} \quad \rho_Y$$
$$(4.37)$$

Recall that

$$\sup_{s \leq t \leq 1} E_{\mu_X}\left(x_t^2\right) < \infty$$

$$\sup_{s \leq t \leq 1} E_{\mu_{X \times Y}}\left[x_t^2 y_t^2\right] < \infty$$

and $f > k > 0$, and g is globally Lipschitz. By Fubini's theorem and the Dominated Convergence theorem it follows that

$$\lim_{M \to \infty} \int_{\Lambda \cap \Omega_M} \psi_u g_u^T f_u^{-1} \, d\mu_X = \int_\Lambda \psi_u g_u^T f_u^{-1} \, d\mu_X \quad \text{a.s.} \quad \mu_Y \qquad (4.38)$$

where we use the continuity in t of the integrand and of the sample paths.

Let

$$f_M = \int_{\Lambda \cap \Omega_M} \psi_u g_u^T f_u^{-1} \, d\mu_X \qquad (4.39)$$

$$f = \lim_{M \to \infty} f_M \qquad (4.40)$$

The sequence

$$\int f_M \, dy_s , \quad \int f \, dy_s \quad M = 1,2,\ldots$$

is a martingale (w.r.t. $B(y_u, \; s \leq u \leq t))$ and using the martingale convergence theorem (Theorem 3.6) we have

$$\lim_{M \to \infty} \int f_M \, dy_s = \int f \, dy_s \qquad (4.41)$$

Therefore

$$\int_\Lambda \int_s^t \psi_u g_u^T f_u^{-1} \, dy_u \, d\mu_X = \int_s^t \int_\Lambda \psi_u g_u^T f_u^{-1} \, d\mu_X \, dy_u \quad \text{a.s.} \quad \mu_Y \qquad (4.42)$$

This was done for fixed t, but using the continuity in t of the stochastic integral we can obtain the result for all $t \in T$. ∎

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We will now derive a stochastic equation for the numerator of the conditional probability density and subsequently a stochastic differential equation for the conditional probability density.

Define the function r as follows

$$r(x,t|\alpha,s,y_u, \ s \leq u \leq t) \ = \ E_{\mu_X}[\psi_t| x_t = x]p(s,\alpha;t,x) \qquad (4.43)$$

Lemma 4.4. For all $t \in (s,1]$ a version of r is given by the following stochastic equation

$$r(x,t|\alpha,s,y_u, \ s \leq u \leq t) \ = \ p_X(s,\alpha;t,x)$$

$$+ \int_s^t \int_{\mathcal{R}^n} p_X\left(u,x';t,x\right)g^T\left(u,x',y_u\right)f^{-1}(u,y_u)r\left(x',u|\alpha,s,y_v, \ s \leq v \leq u\right) dx' \ dy_u$$

$$\qquad (4.44)$$

Proof (cf. Mortensen, Ref. 8).

From the previous lemma (Lemma 4.3)

$$E_{\mu_X}\left[\int_s^t \psi_u g_u^T f_u^{-1} \ dy_u| x_t\right] \ = \ \int_s^t E_{\mu_X}\left[\psi_u g_u^T f_u^{-1}| x_t\right] dy_u \qquad (4.45)$$

Using this result and the stochastic differential rule for ψ_t (cf. Eq. (1.22)) we have

$$E_{\mu_X}[\psi_t| x_t = x] \ = \ 1 + \int_s^t E_{\mu_X}\left[\psi_u g_1^T f_u^{-1}| x_t = x\right] dy_u \qquad (4.46)$$

We will now give a characterization for the integrand of the above stochastic

integral, i.e., $E_{\mu_X}\left[\psi_u g_u^T f_u^{-1} \mid x_t = x\right]$

$$E_{\mu_X}\left[\psi_u g_u^T f_u^{-1} \mid x_t = x\right] = E_{\mu_X}\left[E_{\mu_X}\left[\psi_u g_u^T f_u^{-1} \mid x_t = x, x_u = x'\right] \mid x_t = x\right] \quad \text{a.s.} \quad \mu_X$$
(4.47)

Since we are given $B(y_u, \; s \le u \le t)$

$$E_{\mu_X}\left[\psi_u g_u^T f_u^{-1} \mid x_t = x\right] = E_{\mu_X}\left[g^T f_u^{-1} E_{\mu_X}\left[\psi_u \mid x_u = x', x_t = x\right] \mid x_t = x\right]$$
(4.48)

Since ψ_u is independent of x_t $t > u$ ((ψ_u, x_u, y_u) is a Markov process) we have for the R.H.S.

$$= E_{\mu_X}\left[g_u^T f_u^{-1} E_{\mu_X}\left[\psi_u \mid x_u = x'\right] \mid x_t = x\right]$$

$$= \int_{\mathcal{R}^n} \frac{p_X(u,x';t,x)\, p_X(s,\alpha;u,x')}{p_X(s,\alpha;t,x)}\, g^T(u,x',y_u) f^{-1}(u,y_v) E_{\mu_X}\left[\psi_u \mid x_u = x'\right]\, dx'$$

$$= \frac{1}{p_X(s,\alpha;t,x)} \int_{\mathcal{R}^n} p_X(u,x';t,x)\, g^T(u,x',y_u) f_u^{-1} r\left(x',u \mid \alpha, s, y_v, \; s \le v \le u\right)\, dx'$$
(4.49)

Multiply Eq. (4.49) by $p(s,\alpha;t,x)$ and use Eq. (4.46) which gives the result. To establish the result for all $t \in (s,1]$ we note the continuity in t of p_X and of the stochastic integral. ∎

We shall now derive a differential expression for r. If we formally take the differential of the integral equation for r (Eq. 4.44) we have

$$dr(x,t|\alpha,s,y_u, \ s \le u \le t) \ = \ dp_X(s,\alpha;t,x) + g^T(t,x,y_t)f^{-1}(t,y_t) \ dy_t$$

$$+ \int_s^t \int_{\mathcal{R}^n} dp_X\left(u,x';t,x\right)g^T\left(u,x',y_u\right)f^{-1}(u,y_u)r\left(x',u|\alpha,s,y_v, \ s \le v \le u\right) \ dx' \ dy_u$$

For the differential of the double integral we would expect to obtain

$$g^T(t,x,y_t)f^{-1}(t,y_t) \ dy_t$$

$$+ \int_s^t \int_{\mathcal{R}^n} dp_X\left(u,x';t,x\right)g^T\left(u,x',y_u\right)f^{-1}(u,y_u)r\left(x',u|\alpha,s,y_v, \ s \le v \le u\right) \ dx' \ dy_u$$

Clearly the only difficulty we have is in interchanging the integral and differential operations in the last term.

To justify this last result we first note that

$$E[\varphi_t\psi_t] \ = \ 1$$

Thus by Fubini's theorem we can conclude that

$$E_{\mu_X}[\psi_t] < \infty \qquad \forall t \quad \text{a.s.} \quad \mu_Y$$

Note also that f is strictly positive definite and that

$$\sup_{s \le t \le 1} E_{\mu_{X \times Y}}\left[x_t^2 y_t^2\right] < \infty$$

Since the function r is, except for a normalizing constant, a conditional probability, the stochastic integral

$$\int_s^t \int_{\mathcal{R}^n} g_u^T f_u^{-1} r\left(x',u|\alpha,s,y_v, \ s \le v \le u\right) \ dx' \ dy_u$$

is well defined and can be defined as the limit of step function approximations. Since the terms of dp_X are bounded (cf. the proof of Lemma 2.4) we use the step function approximations together with the Bounded Convergence theorem to conclude

$$d \int_s^t \int_{\mathscr{R}^n} p_X\left(u,x';t,x\right)g^T\left(u,x',y_u\right)f^{-1}(u,y_u)r\left(x',u|\alpha,s,y_v,\ s \leq v \leq u\right)\ dx'\ dy_u$$

$$= g^T(t,x,y_t)f^{-1}(t,y_t)\ dy_t$$

$$+ \int_s^t \int_{\mathscr{R}^n} dp_X\left(u,x';t,x\right)g^T\left(u,x',y_u\right)f^{-1}(u,y_u)r\left(x',u|\alpha,s,y_s,\ s \leq v \leq u\right)\ dx'\ dy_u$$

$$(4.50)$$

In the differential expression for r we can write the terms of dp_X as functions of r and its derivatives by using the integral equation for r (Eq. 4.44) and applying the techniques used to prove Eq. (4.50) to interchange the partial derivatives, that is, we obtain

$$-\sum_{i=1}^n \frac{\partial(a_i(t,x)r)}{\partial x_i} = -\sum_{i=1}^n \frac{\partial(a_i(t,x)p_X(s,\alpha;t,x))}{\partial x_i}$$

$$-\int_s^t \int_{\mathscr{R}^n} \sum_{i=1}^n \frac{\partial\left(a_i(t,x)p_X\left(u,x';t,x\right)\right)}{\partial x_i}\ g^T\left(u,x',y_u\right)$$

$$f^{-1}(u,y_u)r\left(x',u|\alpha,s,y_v,\ s \leq v \leq u\right)\ dx'\ dy_u$$

$$\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2(c_{ij}(t,x)r)}{\partial x_i \partial x_j} = \frac{1}{2}\sum\frac{\partial^2(c_{ij}(t,x)p_X(s,\alpha;t,x))}{\partial x_i \partial x_j}$$

$$+\int_s^t\int_{\mathcal{R}^n}\frac{1}{2}\sum\frac{\partial^2\left(c_{ij}(t,x)p_X\left(u,x';t,x\right)\right)}{\partial x_i \partial x_j}\, g^T\left(u,x',y_u\right)$$

$$f^{-1}(u,y_u)r\left(x',u|\alpha,s,y_v,\ s\leq v\leq u\right)\,dx'\,dy_u$$

We therefore obtain

$$dr(x,t|\alpha,s,y_u,\ s\leq u\leq t) = \left(\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2(c_{ij}(t,x)r)}{\partial x_i \partial x_j}\right.$$

$$\left.-\sum_{i=1}^{n}\frac{\partial(a_i(t,x)r)}{\partial x_i}\right)dt + g^T(t,x,y_t)f^{-1}(t,y_t)r\,dy_t \qquad (4.51)$$

We now consider the denominator term in the expression for the conditional probability density. By our result for interchanging expectation and stochastic integral (Lemma 4.3) and by the continuity of the sample paths we have

$$E_{\mu_X}[\psi_t] = E_{\mu_X}\left[1 + \int_s^t \psi_u g_u^T f_u^{-1}\right]dy_u = 1 + \int_s^t E_{\mu_X}\left[\psi_u g_u^T f_u^{-1}\right]dy_u \quad (4.52)$$

Let

$$q_t = E_{\mu_X}[\psi_t] \qquad (4.53)$$

$$r_t = r(x,t|\alpha,s,y_u,\ s\leq u\leq t) \qquad (4.54)$$

$$p_t = p(x,t|\alpha,s,y_u, \ s \le u \le t) = r_t q_t^{-1} \tag{4.55}$$

We have calculated the differential for q_t^{-1} in an example in Chapter 1 (Eq. 1.27). It is the following

$$d\left(q_t^{-1}\right) = - q_t^{-2}\, dq_t + q_t^{-3} E_{\mu_X}\left[\psi_t g_t^T\right] f_t^{-1} E_{\mu_X}[\psi_t g_t]\, dt \tag{4.56}$$

where

$$dq_t = E_{\mu_X}\left[\psi_t g_t^T\right] f_t^{-1}\, dy_t \tag{4.57}$$

The differential for p_t is

$$dp_t = (dr_t) q_t^{-1} + r_t\left(dq_t^{-1}\right) + g_t^T f_t^{-1} E_{\mu_X}[\psi_t g_t] q_t^{-2} r_t\, dt \tag{4.58}$$

Define $\hat{g}_t$ as

$$\hat{g}_t = \frac{E_{\mu_X}[\psi_t g_t]}{E_{\mu_X}[\psi_t]} \tag{4.59}$$

Therefore

$$dp_t = \left(\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2(c_{ij}(t,x)p_t)}{\partial x_i \partial x_j} - \sum_{i=1}^{n}\frac{\partial(a_i(t,x)p_t)}{\partial x_i}\right)dt$$

$$+ \left(g^T - \hat{g}^T\right)f_t^{-1}\left(dy_t - \hat{g}_t\right)p_t \tag{4.60}$$

This completes the proof of the theorem. ∎

With this equation we therefore have in principle the solution of a fairly general nonlinear filtering problem. This result is analogous to the expressions obtained by Bucy [Ref. 7], Kushner [Ref. 5], Mortensen [Ref. 8] and Shiryaev [Ref. 42].

C. CONDITIONAL MOMENTS

For some problems in filtering theory the diffusion matrix, b, for the state Markov process (Eq. 4.1) does not have an inverse ard the conditional probability density may not exist because the transition density, p_X, may not exist. For such problems we can obtain a stochastic equation for the conditional moments and more generally for smooth functions of the process $\{x_t\}$.

We state our result in the following theorem.

<u>Theorem 4.2.</u> Let $\{x_t\}$ and $\{y_t\}$ satisfy

$$dx(t,\omega) = a(t,x(t,\omega)) \, dt + b(t,x(t,\omega)) \, dB(t,\omega) \qquad (4.61)$$

$$dy(t,\omega) = g(t,x(t,\omega),y(t,\omega)) \, dt + h(t,y(t,\omega)) \, d\tilde{B}(t,\omega) \qquad (4.62)$$

where $x_s = \alpha$, $y_s = 0$, $t \in [s,1]$, $\{B_t\}$ and $\{\tilde{B}_t\}$ are independent Brownian motions in $\mathcal{R}^n$ and $\mathcal{R}^m$ respectively and the drift vector $a(t,x)$ and the diffusion matrix $b(t,x)$ are continuous in t and globally Lipschitz continuous in x, the drift vector $g(t,x,y)$ is continuous in t and globally Lipschitz continuous in x and y and the diffusion matrix $h(t,y)$ is continuous in t and glcbally Lipschitz continuous in y and the symmetric matrix f $\left(f = h^T h\right)$ is strictly positive definite.

Let γ be a real-valued twice continuously differentiable function defined on $\mathcal{R}^n$ such that

$$\iint_{T\ \Omega} |\gamma(x_t)|^2 \, dP \, dt < \infty \qquad (4.63)$$

$$\iint_{T\ \Omega} |x_t \gamma'(x_t)|^2 \, dP \, dt < \infty \qquad (4.64)$$

$$\iint_{T\ \Omega} |x_t^2 \gamma''(x_t)| \, dP \, dt < \infty \qquad (4.65)$$

where the prime denotes differentiation. Then the conditional expectation of γ, $E[\gamma(x_t)|\alpha, s, y_u, \ s \le u \le t]$, denoted as $\hat{\gamma}_t$ satisfies

$$d\hat{\gamma}_t = \widehat{\mathcal{L}\gamma(x_t)} \, dt$$

$$+ (\widehat{\gamma(x_t)g(t,x_t,y_t)} - \widehat{\gamma(x_t)}\widehat{g(t,x_t,y_t)})^T f_t^{-1}(dy_t - \widehat{g(t,x_t,y_t)} \, dt) \quad (4.67)$$

where for example,

$$\widehat{\gamma(x_t)} = \frac{E_{\mu_X}[\gamma(x_t)\psi_t]}{E_{\mu_X}[\psi_t]} \qquad (4.68)$$

$$\mathcal{L}_{t,x}(\cdot) = \sum_i a_i(t,x) \frac{\partial(\cdot)}{\partial x_i} + \frac{1}{2} \sum_{i,j} c_{ij}(t,x) \frac{\partial^2(\cdot)}{\partial x_i \partial x_j} \qquad (4.69)$$

ψ_t is given by Eq. (4.8).

Proof.

By our results for the conditional probability (Lemma 4.1) we have

$$E[r(x_t)|\alpha, s, y_u, \ s \le u \le t] = \frac{E_{\mu_X}[r(x_t)\psi_t]}{E_{\mu_X}[\psi_t]} \qquad (4.70)$$

As in our derivation of the conditional probability density we first consider the differential of the numerator of the R.H.S. of the above equation. We apply the stochastic differential rule to $r(x_t)\psi_t$ to obtain

$$r(x_t)\psi_t - r(x_s)\psi_s = \int_s^t \psi_u r(x_u) g_u^T f_u^{-1} \ dy_u + \int_s^t \mathcal{L}r(x_u)\psi_u \ du$$

$$+ \int_s^t \psi_u r'(x_u) b_u \ dB_u \qquad (4.71)$$

With our assumption that

$$\iint_{T\,\Omega} |x_t r'(x_t)|^2 \ dP \ dt < \infty$$

and the fact that ψ_u is a density we have for almost all $y_u, \ s \le u \le t$

$$E_{\mu_X}\left[\int_s^t \psi_u r'(x_u) b_u \ dB_u\right] = 0 \qquad (4.72)$$

We have only to prove the following

$$E_{\mu_X} \int_s^t \mathcal{L}r(x_u)\psi_u \, du \;=\; \int_s^t E_{\mu_X} \mathcal{L}r(x_u)\psi_u \, du \qquad (4.73)$$

$$E_{\mu_X} \int_s^t \psi_u r(x_u) g_u^T f_u^{-1} \, dy_u \;=\; \int_s^t E_{\mu_X} \psi_u r(x_u) g_u^T f_u^{-1} \, dy_u \qquad (4.74)$$

to obtain our result.

The first equality (Eq. 4.73) follows directly from Fubini's theorem with the integrability assumptions made. The second result is more difficult but the proof is quite similar to the proof of Lemma 4.3. We state the result in the following lemma.

Lemma 4.5. The following interchange of order of integrations is valid a.s.

$$\int_\Omega \int_s^t \psi_u r(x_u) g_u^T f_u^{-1} \, dy_u \, d\mu_X \;=\; \int_s^t \int_\Omega \psi_u r(x_u) g_u^T f_u^{-1} \, d\mu_X \, dy_u \qquad (4.75)$$

Proof.

Let

$$\Omega_M \;=\; \{\omega : \sup_{s \le t \le 1} |x_t| < M , \quad \sup_{s \le t \le 1} |y_t| < M\} \qquad (4.76)$$

Since the processes $\{x_t\}$ and $\{y_t\}$ have continuous sample paths we have

$$P\left(\bigcup_M \Omega_M\right) \;=\; 1 \;=\; \lim_{M \to \infty} P(\Omega_M) \qquad (4.77)$$

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The last equality follows since $\Omega_{M+1} \supset \Omega_M$. By the boundedness of the integrand on Ω_M we can define the stochastic integral as a limit of step function approximations on this set. The interchange of order of integrations is clearly valid for step functions since we are given $B(y_u, \ s \leq u \leq t)$. Therefore we have

$$\int_{\Omega_M} \int_s^t \psi_u \gamma(x_u) g_u^T f_u^{-1} \, dy_u \, d\mu_X = \int_s^t \int_{\Omega_M} \psi_u \gamma(x_u) g_u^T f_u^{-1} \, d\mu_X \, dy_u \qquad (4.78)$$

Since $\gamma(x_u)$ is assumed square integrable on the product measure $du \times d\mu_X$ and the other terms correspond to the density function we have

$$\lim_{M \to \infty} \int_{\Omega_M} \int_s^t \psi_u \gamma(x_u) g_u^T f_u^{-1} \, dy_u \, d\mu_X = \int_\Omega \int_s^t \psi_u \gamma(x_u) g_u^T f_u^{-1} \, dy_u \, d\mu_X \qquad (4.79)$$

Similarly we can show

$$\lim_{M \to \infty} \int_{\Omega_M} \psi_u \gamma(x_u) g_u^T f_u^{-1} \, d\mu_X = \int_\Omega \psi_u \gamma(x_u) g_u^T f_u^{-1} \, d\mu_X \qquad (4.80)$$

We can now obtain the following equality

$$\lim_{M \to \infty} \int_{\Omega_M} \int_s^t \psi_u \gamma(x_u) g_u^T f_u^{-1} \, dy_u \, d\mu_X = \lim_{M \to \infty} \int_s^t \int_{\Omega_M} \psi_u \gamma(x_u) g_u^T f_u^{-1} \, d\mu_X \, dy_u \qquad (4.81)$$

since the limit on the L.H.S. is well defined.

By the uniform integrability of the sequence $\langle f_M \rangle$

$$f_M = \int_{\Omega_M} \psi_u \gamma(x_u) g_u^T f_u^{-1} \, d\mu_X \qquad (4.82)$$

we have

$$\lim_{M \to \infty} \int_s^t \int_{\Omega_M} \psi_u \gamma(x_u) g_u^T f_u^{-1} \, d\mu_X \, dy_u = \int_s^t \lim_{M \to \infty} \int_{\Omega_M} \psi_u \gamma(x_u) g_u^T f_u^{-1} \, d\mu_X \, dy_u \qquad (4.83)$$

Therefore we have

$$\int_s^t \int_\Omega \psi_u \gamma(x_u) g_u^T f_u^{-1} \, d\mu_X \, dy_u = \int_\Omega \int_s^t \psi_u \gamma(x_u) g_u^T f_u^{-1} \, dy_u \, d\mu_X \qquad (4.84)$$

Recall Eq. (4.56) where we have already calculated the differential for

$$\frac{1}{E_{\mu_X}[\psi_t]}$$

Combining this differential with our differential result for the numerator term $E_{\mu_X}[\gamma(x_t)\psi_t]$ we have

$$d\hat{\gamma}_t = \widehat{\mathscr{L}\gamma}(x_t) \, dt$$

$$+ (\widehat{\gamma(x_t)g(t,x_t,y_t)} - \widehat{\gamma(x_t)}\widehat{g(t,x_t,y_t)})^T f_t^{-1}(dy_t - \widehat{g(t,x_t,y_t)} \, dt) \qquad (4.85)$$

where for example,

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$$\widehat{\gamma(x_t)} = \frac{E_{\mu_X}[\gamma(x_t)\psi_t]}{E_{\mu_X}[\psi_t]} \qquad\qquad (4.86)$$

This completes the proof of the theorem. ∎

V. EVALUATION OF LIKELIHOOD FUNCTIONS

Some results for the detection of a stochastic signal in white Gaussian noise have been obtained where the likelihood function (Radon-Nikodym derivative) could be calculated recursively from a differential equation. The terms of this differential equation were related to quantities that arise in filtering problems. Schweppe [Ref. 15] considered the detection problem where the signal was generated by white noise into a finite dimensional linear system and formally showed the relation of a differential equation for the likelihood function to the linear filtering results of Kalman and Bucy. Sosulin and Stratonovich [Ref. 17] considered the detection problem where the signal was a diffusion process and formally related a differential equation for the likelihood function to the non-linear filtering problem.

In this chapter we shall rigorously derive a stochastic differential equation for the likelihood function of a stochastic signal (diffusion process) in white Gaussian noise and relate this to the results of Schweppe and Sosulin and Stratonovich. We shall also discuss detection problems of a stochastic signal (diffusion process) in correlated noise (diffusion process) obtaining, for some problems, necessary and sufficient conditions for nonsingular detection. For the nonsingular case we derive a differ-ential equation for the evolution of the likelihood function by relating the detection problem to a detection problem with white noise.

A. A DIFFUSION PROCESS IN WHITE NOISE

We consider the two hypotheses detection problem of a stochastic signal (diffusion process) in white Gaussian noise described by the following stochastic differential equations

$$dy_t = H(t)x_t \, dt + d\tilde{B}_t \quad \text{for} \quad \theta = 1$$

$$= d\tilde{B}_t \quad \text{for} \quad \theta = 0 \qquad (5.1)$$

where

$$dx_t = a(t,x_t) \, dt + b(t,x_t) \, dB_t \qquad (5.2)$$

and $t \in [s,1]$. We furthermore assume

1. The initial conditions are $x(s) = \alpha$, $y(s) = 0$ and x_t is an n vector, y_t is an m vector, $H(t)$ is an $m \times n$ matrix continuous in t and B_t and $\tilde{B}_t$ are independent n and m dimensional Brownian motions respectively. The drift vector $a(t,x)$ and the diffusion matrix $b(t,x)$ are continuous in t and globally Lipschitz continuous in x. The prior probabilities are $P(\theta = 1) = \Pi_1$ and $P(\theta = 0) = \Pi_0$.

1. A Differential Equation for the Likelihood Function

We define the following measures as we did for the nonlinear filtering problem. The measure μ_X is the measure induced on the space $C_n[s,1]$ by the solution of

$$dx(t,\omega) = a(t,x(t,\omega)) \, dt + b(t,x(t,\omega)) \, dB(t,\omega) \qquad (5.2)$$

where $t \in [s,1]$ and $x(s) = \alpha$. The measure ρ_Y is the Wiener measure induced on $C_m[s,1]$ by the solution of

$$dy(t,\omega) = d\tilde{B}(t,\omega) \qquad (5.3)$$

where $t \in [s,1]$ and $y(s) = 0$. The measure μ_{XxY} is the measure induced on the space $C_{n+m}[s,1]$ by the solutions of

$$dx(t,\omega) = a(t,x(t,\omega)) \, dt + b(t,x(t,\omega)) \, dB(t,\omega) \qquad (5.2)$$

$$dy(t,\omega) = H(t)x(t,\omega) \, dt + d\tilde{B}(t,\omega) \qquad (5.4)$$

where $t \in [s,1]$, $x(s) = \alpha$ and $y(s) = 0$.

We define the function ψ_t as

$$\psi_t(x_u,y_u, \; s \le u \le t) = \exp\left[\int_s^t x_u^T H_u^T \, dy_u - \frac{1}{2}\int_s^t x_u^T H_u^T H_u x_u \, du\right] \qquad (5.5)$$

where

$$\psi_1 = \frac{d\mu_{XxY}}{d(\mu_X \times \rho_Y)} \qquad (5.6)$$

We now derive an expression for the likelihood function for the detection problem (Eq. 5.1).

Lemma 5.1. The likelihood function, Λ_t, for the detection problem, Eq. (5.1), where we assume H is given by

$$\Lambda_t = E_{\mu_X}[\psi_t] \qquad (5.7)$$

where ψ_t is given by Eq. (5.5).

Proof.

The likelihood function, Λ_t, is the Radon-Nikodym derivative of the measures, say ρ_1 and ρ_0, corresponding to the two hypotheses

$\theta = 1$ and $\theta = 0$, i.e.,

$$\Lambda_t = \frac{d\rho_1}{d\rho_0} \tag{5.8}$$

where the measures are functionals of the sample paths to time t. Let $\Gamma \in B(y_u, \ s \leq u \leq t)$. Then

$$\rho_1(\Gamma) = \int\int_\Gamma \psi_t \ d(\mu_X \times \rho_0)$$

$$= \int_\Gamma E_{\mu_X}[\psi_t] \ d\rho_0 \tag{5.9}$$

By definition of the Radon-Nikodym derivative we have

$$\Lambda_t = E_{\mu_X}[\psi_t] \qquad \blacksquare \tag{5.10}$$

With an expression for the likelihood function we are able to derive a stochastic equation for a monotonic function of the likelihood function.

Theorem 5.1. Consider the two hypotheses detection problem

$$dy_t = H(t)x_t \ dt + d\tilde{B}_t \qquad \text{for} \quad \theta = 1$$

$$= \qquad\qquad d\tilde{B}_t \qquad \text{for} \quad \theta = 0 \tag{5.1}$$

where we assume H. Let Λ_t be the likelihood function for this detection problem.

Then the process $\{z_t\}$ defined as

$$z_t = \ln \Lambda_t \tag{5.11}$$

satisfies the following stochastic differential equation

$$dz_t = \hat{x}_t^T H_t^T \, dy_t - \frac{1}{2} \hat{x}_t^T H_t^T H_t \hat{x}_t \, dt \qquad (5.12)$$

where

$$\hat{x}_t = \frac{E_{\mu_X}[\psi_t x_t]}{E_{\mu_X}[\psi_t]} \qquad (5.13)$$

is the conditional mean of x_t given $\{y_u, \ s \leq u \leq t\}$ (cf. Eq. 4.67).

Proof.

We apply the stochastic differential rule (Theorem 1.2) to

$$z_t = \ln \Lambda_t \qquad (5.11)$$

and obtain

$$dz_t = \frac{d\Lambda_t}{\Lambda_t} - \frac{1}{2} \frac{(d\Lambda_t)^2}{\Lambda_t^2} \qquad (5.14)$$

The terms $d\Lambda_t$ and $(d\Lambda_t)^2$ have been calculated in an example of the stochastic differential rule in Chapter 1 (Eq. 1.27) and are

$$d\Lambda_t = E_{\mu_X}\left[\psi_t x_t^T H_t^T\right] dy_t \qquad (5.15)$$

$$(d\Lambda_t)^2 = E_{\mu_X}\left[\psi_t x_t^T H_t^T\right] E_{\mu_X}[\psi_t H_t x_t] \, dt \qquad (5.16)$$

Therefore we have

$$dz_t = \hat{x}_t^T H_t^T \, dy_t - \frac{1}{2} \hat{x}_t^T H_t^T H_t \hat{x}_t \, dt \qquad (5.12)$$

where

$$\hat{x}_t = \frac{E_{\mu_X}[\psi_t x_t]}{E_{\mu_X}[\psi_t]} \qquad \blacksquare \qquad (5.13)$$

2. Comparison With Previous Results

The stochastic differential equation for $\{z_t\}$ (Eq. 5.12) is different in appearance from the equation

$$dz_t = \hat{x}_t^T H_t^T \, dy_t - \frac{1}{2} \widehat{x_t^T H_t^T H_t x_t} \, dt \qquad (5.17)$$

where

$$\widehat{x_t^T H_t^T H_t x_t} = \frac{E_{\mu_X}\left[\psi_t x_t^T H_t^T H_t x_t\right]}{E_{\mu_X}[\psi_t]} \qquad (5.18)$$

obtained by Schweppe [Ref. 15] and Sosulin and Stratonovich [Ref. 17]. This anomaly between the two equations for the likelihood function occurs because of the different definitions of stochastic integral and the convergence properties of discrete time versions of a stochastic equation [Refs. 9,43,44,45]. In Eq. (5.17), the "correlator" term (stochastic integral) has to be interpreted in the sense of Stratonovich [Ref. 9] while in our equation (Eq. 5.12) the stochastic integral has the K. Itô interpretation.

To clarify the relation between these two results we will indicate the correction term. If we perform an integration by parts on the stochastic integral

$$\int_{s}^{l} \hat{x}_t^T H_t^T \, dy_t \tag{5.19}$$

(this integration by parts is valid since this stochastic integral can be defined as the limit of step function approximations, $\hat{x}_t$ has a differential given by Eq. (5.24), and we assume dH/dt exists and is continuous) then the stochastic integral we obtain is

$$-\int_{s}^{l} y_t^T H_t \, d\hat{x}_t \tag{5.20}$$

We shall consider the case of scalar observations $(m = 1)$, and compute the correction term first for the Gaussian case.

Consider the K. Itô stochastic integral

$$\int \Phi(t, y_t) \, dy_t \tag{5.21}$$

where

$$dy_t = g(t, y_t) \, dt + d\tilde{B}_t \tag{5.22}$$

Then if we add the term (cf. Ref. 9)

$$\frac{1}{2} \int \frac{\partial \Phi(t, y_t)}{\partial y} \, dt \tag{5.23}$$

to the stochastic integral we have the integral described by Stratonovich.

Recalling the stochastic differential equation for $\hat{x}_t^{(i)}$, a component of $\hat{x}_t$

$$d\hat{x}_t^{(i)} = \mathscr{L}x_t^{(i)} \, dt + \left(\widehat{x_t^{(i)}H_t x_t} - x_t^{(i)}H_t\hat{x}_t\right)^T\left(dy_t - H\hat{x}_t \, dt\right) \quad (5.24)$$

and that in the linear case [Ref. 1] the function

$$\widehat{x_t^{(i)}H_t x_t} - \hat{x}_t^{(i)}H_t\hat{x}_t \quad (5.25)$$

depends only on t, the correction term which we must add to our differential equation is

$$-\frac{1}{2}\int \left(\widehat{x_t^T H_t^T H_t x_t} - \hat{x}_t^T H_t^T H_t \hat{x}_t\right) dt \quad (5.26)$$

which gives Schweppe's result.

To relate our result to the equation obtained by Sosulin and Stratonovich we heuristically describe a "generalized" Stratonovich integral motivated by the fact that we want the rules of transformation of ordinary calculus to apply to this integral. We use a result of K. Itô [Ref. 30] for the product of stochastic integrals

$$\int_u^v \zeta(t,\omega) \, dB(t,\omega) \int_u^v \eta(s,\omega) \, dB(s,\omega) = \int_u^v \zeta(t,\omega) \int_u^t \eta(s,\omega) \, dB(s,\omega) \, dB(t,\omega)$$

$$+ \int_u^v \eta(s,\omega) \int_u^s \zeta(t,\omega) \, dB(t,\omega) \, dB(s,\omega) + \int_u^v \zeta(t,\omega)\eta(t,\omega) \, dt \quad (5.27)$$

whereas if these were ordinary integrals we would not have the last term. We therefore "split" this ordinary integral between the two stochastic integrals as

$$\int\limits_{u}^{V} \zeta(t,\omega) \int\limits_{u}^{t} \eta(s,\omega) \, dB(s,\omega) \, dB(t,\omega) + \frac{1}{2} \int\limits_{u}^{V} \zeta(t,\omega)\eta(t,\omega) \, dt$$

$$+ \int\limits_{u}^{v} \eta(s,\omega) \int\limits_{u}^{s} \zeta(t,\omega) \, dB(t,\omega) \, dB(s,\omega) + \frac{1}{2} \int\limits_{u}^{V} \zeta(t,\omega)\eta(t,\omega) \, dt$$

Defining this as a Stratonovich integral we then have our correction term which reduces to the correction term (Eq. 5.23) for the diffusion process case. This correction term again is

$$- \frac{1}{2} \int \left(\widehat{x_t^T H_t^T H_t x_t} - \hat{x}_t^T H_t^T H_t \hat{x}_t \right) dt \qquad (5.28)$$

from Eq. (5.24). Therefore with this term our results correspond with the equation of Sosulin and Stratonovich.

B. A DIFFUSION PROCESS IN CORRELATED NOISE

Having established a stochastic differential equation for the likelihood function for the detection of a stochastic signal (diffusion process) in white Gaussian noise the obvious extension of these results is to the detection of a stochastic signal (diffusion process) in correlated noise (diffusion process). We describe a problem of this type by the following stochastic equations

$$y_t = H(t)x_t + z_t \qquad \text{for} \ \theta = 1$$

$$= z_t \qquad \text{for} \ \theta = 0 \qquad (5.29)$$

where $t \in [s,1]$ and

$$dx_t = a(t, x_t)\, dt + b(t, x_t)\, dB_t \qquad (5.30)$$

$$dz_t = g(t, z_t)\, dt + h(t)\, d\tilde{B}_t \qquad (5.31)$$

We furthermore assume

$\tilde{H}$. The initial conditions are $x(s) = \alpha$, $z(s) = 0$ and x_t is an

n vector, z_t is an m vector, $H(t)$ is an m x n matrix that is

continuous in t and B_t and $\tilde{B}_t$ are independent n and m

dimensional Brownian motions respectively. The drift vectors $a(t, x)$

and $g(t, x)$ are continuous in t and globally Lipschitz continuous

in x. The diffusion matrix $b(t, x)$ is continuous in t and globally

Lipschitz continuous in x. The diffusion matrix $h(t)$ is continuous

in t and h^{-1} exists. The prior probabilities are $P(\theta = 1) = \Pi_1$

and $P(\theta = 0) = \Pi_0$. The derivative of $H(t)$, $H'(t)$, is continuous in t.

1. Necessary and Sufficient Conditions for Nonsingular Detection

We shall now obtain necessary and sufficient conditions for non-

singular detection for the problem described above (Eq. 5.29).

Theorem 5.2. Consider the detection problem

$$y_t = H(t)x_t + z_t \qquad \text{for } \theta = 1$$

$$\qquad (5.29)$$

$$= \qquad z_t \qquad \text{for } \theta = 0$$

where $t \in [s, 1]$ and

$$dx_t = a(t,x_t)\,dt + b(t,x_t)\,dB_t \tag{5.30}$$

$$dz_t = g(t,z_t)\,dt + h(t)\,d\tilde{B}_t \tag{5.31}$$

and we furthermore assume $\tilde{H}$. For this detection problem to be non-singular it is necessary and sufficient that

$$H(t)b(t,x_t) \equiv 0 \tag{5.32}$$

<u>Proof (Sufficiency)</u>.

It will be convenient to change the form of the above detection problem by describing the hypotheses by stochastic differential equations as

$$dy_t = H(t)a(t,x_t)\,dt + H(t)b(t,x_t)\,dB_t + g(t,y_t - H(t)x_t)\,dt$$

$$+ H'(t)x_t\,dt + h(t)\,d\tilde{B}_t \qquad \text{for } \theta = 1 \tag{5.33}$$

$$= \qquad g(t,y_t)\,dt + h(t)\,d\tilde{B}_t \quad \text{for } \theta = 0$$

Now let $H(t)b(t,x_t) \equiv 0$. Then the two hypotheses are

$$dy_t = H(t)a(t,x_t)\,dt + g(t,y_t - H(t)x_t)\,dt + h(t)\,d\tilde{B}_t \quad \text{for } \theta = 1$$

$$+ H'(t)x_t\,dt$$

$$= \qquad g(t,y_t)\,dt + h(t)\,d\tilde{B}_t \qquad \text{for } \theta = 0$$

Since the process $\{x_t\}$ is generated by $\{B_t\}$ and h^{-1} exists we can easily modify Girsanov's Theorem (Theorem 1.3) to show that

$$\mu_{X \times Y} \ll \mu_X \times \rho_Y \qquad (5.34)$$

where the measure ρ_Y is the measure induced on $C_m[s,1]$ by the solution of

$$dy_{\cdot} = g(t,y_t)\, dt + h(t)\, d\tilde{B}_t \qquad (5.35)$$

for $t \in [s,1]$ and $y(s) = 0$. The measure μ_X is the measure induced on $C_n[s,1]$ by the solution of

$$dx_t = a(t,x_t)\, dt + b(t,x_t)\, dB_t \qquad (5.36)$$

for $t \in [s,1]$ and $x(s) = \alpha$. The measure $\mu_{X \times Y}$ is the measure induced on $C_{n+m}[s,1]$ by the solutions of

$$dx_t = a(t,x_t)\, dt + b(t,x_t)\, dB_t \qquad (5.36)$$

$$dy_t = H(t)a(t,x_t)\, dt + g(t,y_t - H(t)x_t)\, dt + h(t)\, d\tilde{B}_t \qquad (5.37)$$

$$+ H'(t)x_t\, dt$$

for $t \in [s,1]$ and $x(s) = \alpha,\ y(s) = 0$.

Define ψ_t as the density function so that

$$\psi_1 = \frac{d\mu_{X \times Y}}{d(\mu_X \times \rho_Y)} \qquad (5.38)$$

The form of this function ψ_t can be determined from Girsanov's theorem (Theorem 1.3).

The likelihood function, Λ_t, follows from Lemma 5.1 as

$$\Lambda_t = E_{\mu_X}[\psi_t] \tag{5.39}$$

Proof (Necessity).

If b^{-1} exists we can show that we have a singular detection problem by using our result from Chapter 3 (Theorem 3.1) for the absolute continuity of measures corresponding to solutions of stochastic differential equations. Define the measure ρ_X as the measure induced on $C_n[s,1]$ by the solution of

$$dx(t,\omega) = b(t,x(t,\omega))\, dB(t,\omega) \tag{5.40}$$

for $t \in [s,1]$ and $x(s) = \alpha$, the measure $\rho_{X\times Y}$ is the measure induced on $C_{n+m}[s,1]$ by the solutions of

$$dx(t,\omega) = b(t,x(t,\omega))\, dB(t,\omega) \tag{5.40}$$

$$dy(t,\omega) = H(t)b(t,x(t,\omega))\, dB(t,\omega) + h(t)\, d\tilde{B}(t,\omega) \tag{5.41}$$

for $t \in [s,1]$ and $x(s) = \alpha,\ y(s) = 0$, and the measure $\mu_{X\times Y}$ is the measure induced on $C_{n+m}[s,1]$ by the solutions of

$$dx_t = a(t,x_t)\, dt + b(t,x_t)\, dB_t \tag{5.42}$$

$$dy_t = H(t)a(t,x_t)\, dt + H(t)b(t,x_t)\, dB_t + g(t,y_t - H(t)x_t)\, dt + h(t)\, d\tilde{B}_t$$
$$+ H'(t)x_t\, dt \tag{5.43}$$

The measure ρ_Y is defined by Eq. (5.35).

By Theorem 3.1 it follows that

$$\rho_{X \times Y} \not< \rho_X \times \rho_Y$$

and

$$\mu_X \equiv \rho_X$$

therefore

$$\mu_{X \times Y} \not< \mu_X \times \rho_Y$$

If b^{-1} does not exist we apply a result due to Wong and Zakai [Ref. 46] stated in the following lemma.

Lemma 5.2. Let $\{x_t\}$ satisfy

$$dx_t = \phi(t,x_t)\, dt + \Gamma(t,x_t)\, dB_t \tag{5.44}$$

where $t \in [s,1]$, $x(s) = \alpha$ and $\phi(t,x)$ and $\Gamma(t,x)$ are continuous in t and globally Lipschitz continuous in x; x_t is an n vector and B_t is n dimensional Brownian motion.

Then

$$\lim_{n \to \infty} \sum_{i=1}^{k(n)-1} \left(x_j\left(t_{i+1}^{(n)}\right) - x_j\left(t_i^{(n)}\right) \right)^2 = \sum_{r=1}^{n} \int_s^1 r_{jr}^2(t,x_t)\, dt \tag{5.45}$$

in the mean where $\Gamma = \{\gamma_{ij}\}$.

For the proof of this lemma the reader is referred to Wong and Zakai [Ref. 46].

Using Eq. (5.45) as our test statistic on the observations we are

able to determine if the signal is present or not and therefore have singular detection. ▮

2. Reduction to a White Noise Detection Problem

In the nonsingular correlated noise detection problem we have for the two hypotheses

$$dy_t = H(t)a(t,x_t)\,dt + g(t,y_t - H(t)x_t)\,dt + h(t)\,d\tilde{B}_t \quad \text{for } \theta = 1$$

$$+ H'(t)x_t\,dt \tag{5.46}$$

$$= g(t,y_t)\,dt + h(t)\,d\tilde{B}_t \quad \text{for } \theta = 0$$

As we mentioned in the proof of Theorem 5.2 an expression for the likelihood function (Eq. 5.39) can be calculated using Girsanov's Theorem (Theorem 1.3). We shall now show how to obtain a differential equation for the likelihood function by considering the detection problem as a white noise type of detection problem.

In the proof of Theorem 5.2 we defined the measures ρ_Y, μ_X, and μ_{XxY} (cf. Eqs. 5.35, 5.36, 5.37). We now define the measure $\rho_{\tilde{Y}}$ induced on $C_m[s,1]$ by the solution of

$$dy_t = h(t)\,d\tilde{B}_t \tag{5.47}$$

for $t \in [s,1]$ and $y(s) = 0$. By Girsanov's Theorem (Theorem 1.3) we have

$$\rho_Y \ll \rho_{\tilde{Y}}$$

The likelihood function, Λ_t, with respect to the measure $\rho_{\underset{\sim}{Y}}$ is

$$\Lambda_t = E_{\mu_X}[\tilde{\psi}_t] \tag{5.48}$$

where $\tilde{\psi}_t$ is determined from ψ_t (Eq. 5.38) and $d\rho_Y/d\rho_{\underset{\sim}{Y}}$. This result follows from the definition of the likelihood function (Eq. 5.8).

We define z_t as

$$z_t = \ln \Lambda_t \tag{5.49}$$

and take the differential of z_t (using Eq. 5.14) to obtain a differential expression for z_t. We therefore have a white noise type of detection problem. Since the details are straightforward they will be omitted here.

3. Some Generalizations

We now consider a more general correlated noise detection problem with the hypotheses

$$
\begin{aligned}
y_t &= H(t)x_t + G(t)z_t \quad \text{for } \theta = 1 \\
&= \qquad\quad G(t)z_t \quad \text{for } \theta = 0
\end{aligned}
\tag{5.50}
$$

where G is a $k \times m$ matrix continuous in t and H is a $k \times n$ matrix continuous in t. The derivatives of H and G, H' and G', are continuous in t. We also assume $\tilde{H}$. We shall make additional assumptions as we proceed.

For this correlated noise detection problem we shall discuss conditions for nonsingular detection. This discussion will be somewhat more informal than the discussion in the above section because we primarily want to

indicate the methods. We again convert our hypotheses to stochastic differential equations

$$dy_t = H(t)a(t,x_t) \, dt + H(t)b(t,x_t) \, dB_t + G(t)g(t,z_t) \, dt$$

$$+ G(t)h(t) \, d\tilde{B}_t \qquad \text{for} \quad \theta = 1$$

$$= G(t)g(t,z_t) \, dt + G(t)h(t) \, d\tilde{B}_t \quad \text{for} \quad \theta = 0$$

$$(5.51)$$

Primarily for simplicity of discussion we shall assume that the observations are scalar, i.e., $k = 1$.

To determine singularity or nonsingularity of the detection problem we shall consider a few cases. If $Hb \not\equiv 0$ then the detection problem is singular by Lemma 5.2. If $Hb \equiv 0$ and $Gh \neq 0$ then the detection problem is nonsingular by applying Girsanov's Theorem (Theorem 1.3). If $Hb \equiv 0$ and $Gh \equiv 0$ we have the hypotheses

$$dy_t = H(t)a(t,x_t) \, dt + G(t)g(t,z_t) \, dt \quad \text{for} \quad \theta = 1$$

$$= G(t)g(t,z_t) \, dt \quad \text{for} \quad \theta = 0$$

Let

$$y_t' = \frac{dy_t}{dt} \qquad (5.52)$$

Then

$$y_t' = H(t)a(t,x_t) + G(t)g(t,z_t) \quad \text{for} \quad \theta = 1$$

$$= G(t)g(t,z_t) \quad \text{for} \quad \theta = 0$$

(5.53)

We shall again convert these hypotheses to stochastic differential equations
by applying the stochastic differential rule (Theorem 1.2). Here we assume
the appropriate differentiability of the functions H, a, G, and g to
apply the differential rule. The stochastic differential equations that
we obtain are symbolically

$$dy_t' = \left(\frac{\partial(Ha)}{\partial t} + H \frac{\partial a}{\partial x} a + \frac{1}{2} H \frac{\partial^2 a}{\partial x^2} b^T b \right) dt + H \frac{\partial a}{\partial x} b \, dB_t$$

$$+ \left(\frac{\partial(Gg)}{\partial t} + G \frac{\partial g}{\partial z} g + \frac{1}{2} G \frac{\partial^2 g}{\partial z^2} h^T h \right) dt + G \frac{\partial g}{\partial z} h \, d\tilde{B}_t \quad \text{for} \quad \theta = 1$$

$$= \left(\frac{\partial(Gg)}{\partial t} + G \frac{\partial g}{\partial z} g + \frac{1}{2} G \frac{\partial^2 g}{\partial z^2} h^T h \right) dt + G \frac{\partial g}{\partial z} h \, d\tilde{B}_t \quad \text{for} \quad \theta = 0$$

(5.54)

We again consider some cases. If

$$H \frac{\partial a}{\partial x} b \neq 0$$

(5.55)

where

$$\frac{\partial a}{\partial x} = \left\{ \frac{\partial a_i(t,x)}{\partial x_j} \right\} = \{\alpha_{ij}\}$$

(5.56)

then the detection problem is singular by Lemma 5.2. If

$$H \frac{\partial a}{\partial x} b + H'b \equiv 0 \qquad (5.57)$$

and

$$G \frac{\partial g}{\partial z} h + G'h \neq 0 \qquad (5.58)$$

where

$$\frac{\partial g}{\partial z} = \left\{ \frac{\partial g_i(t,z)}{\partial z_j} \right\} = \{r_{ij}\} \qquad (5.59)$$

and if the nonzero terms of Eq. (5.59) are not a function of z_t, i.e., these terms arise from linear terms of g then the detection problem is nonsingular. If the nonzero terms of Eq. (5.59) are functions of z_t then the detection problem will be singular by applying Lemma 5.2. If

$$H \frac{\partial a}{\partial x} b + H'b \equiv 0 \qquad (5.60)$$

and

$$G \frac{\partial g}{\partial z} h + G'h \equiv 0 \qquad (5.61)$$

then we have the two hypotheses

$$dy'_t = \left(\frac{\partial(Ha)}{\partial t} + H \frac{\partial a}{\partial x} a + \frac{1}{2} H \frac{\partial^2 a}{\partial x^2} b^T b + H''x_t + H'a \right) dt$$

$$+ \left(\frac{\partial(Gg)}{\partial t} + G \frac{\partial g}{\partial z} g + \frac{1}{2} G \frac{\partial^2 g}{\partial z^2} h^T h + G''z_t + G'g \right) dt \quad \text{for } \theta = 1$$

$$dy_t' = \left(\frac{\partial (Gg)}{\partial t} + G\frac{\partial g}{\partial z}g + \frac{1}{2}G\frac{\partial^2 g}{\partial z^2}h^T h + G''z_t + G'g \right)dt \quad \text{for } \theta = 0$$

$$\tag{5.62}$$

Let

$$y_t'' = \frac{dy_t'}{dt} \tag{5.63}$$

and we proceed as we did with y_t' by taking the differential of y_t''. With appropriate differentiability assumptions it seems reasonable that this procedure will terminate in a finite number of steps, though to prove this finite termination seems difficult. We have mainly included this discussion to indicate the methods to be used to determine if a detection problem is well posed, i.e., nonsingular.

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13. ABSTRACT

Some problems in the filtering and the detection of diffusion processes that are solutions of stochastic differential equations are studied.

Transition densities for Markov process solutions of a large class of stochastic differential equations are shown to exist and to satisfy Kolmogorov's equations. These results extend previously known results by allowing the drift coefficient to be unbounded. With these results for transition densities the nonlinear filtering problem is discussed and the conditional probability of the state vector of the system conditioned on all the past observations is shown to exist and a stochastic equation is derived for the evolution in time of the conditional probability density. A stochastic differential equation is also obtained for the conditional moments. These derivations use directly the continuous time processes.

Necessary conditions that coincide with the previously known sufficient conditions for the absolute continuity of measures corresponding to solutions of stochastic differential equations are obtained. Applications are made to the detection of one diffusion process in another. Previous results on the relation between detection and filtering problems are rigorously obtained and extended.

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